# A Hands-On Introduction To Manifolds and Vector Bundles

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Dedicated to Zahra

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# Preface

Studying manifolds and vector bundles is, in many ways, doing calculus in its most refined and serious form. Over the past century, the foundations of this subject have been firmly established, and a wide array of texts now explore these ideas at varying levels of depth and sophistication.

This book grew out of several graduate courses I have taught at the University of Iowa. While many excellent resources exist and have influenced this book, I have often found it difficult to recommend a single reference that presents the essential ideas in a coherent order, at a measured pace, with enough illustrative examples – all within a manageable length.

The structure of the book reflects its classroom origins. It is formatted for a one-semester course and is organized as a sequence of lectures, each designed to cover a natural chunk of material suitable for one or two sessions. To some extent, the format and selection of topics are aimed at students preparing for PhD qualification exams. The intended audience includes primarily first-year graduate students, though the material is also accessible to advanced undergraduates with a good grasp on real analysis, point-set topology, multivariable calculus, and linear algebra. Thanks to the inclusion of detailed solutions to all exercises, the book is also suitable for independent study.

Throughout, I have placed strong emphasis on examples and computations. Abstract definitions are consistently followed by concrete calculations and carefully chosen exercises, designed to help readers internalize key ideas and prepare for more advanced work. I have repeatedly observed that some students can learn and recite theorems – even reproduce their proofs – yet struggle with applying concepts in explicit computations. For this reason, all exercises in the book are accompanied by detailed solutions. While some books offer a large number of exercises, many of which involve results not covered in the main text, here the exercises are primarily computational and closely tied to the core material.

While the book contains many proofs, they are included only when relevant to understanding the broader framework of manifolds and vector bundles. Local analytic results such as the inverse function theorem or Sard's theorem are used but not proved, as they are standard in real analysis and do not rely on the global structure central to this text. The style of exposition is precise and abstract, with decent use of geometric pictures wherever it helps with digesting the materials.

Unlike texts that front-load extensive background material before introducing manifolds and vector bundles, this book integrates the necessary tools from analysis, topology, calculus, and linear algebra as they arise. This approach allows readers to enter the subject more directly, encountering foundational results in the context where they are needed. For certain results whose proofs are not central to the book's conceptual development, I have provided explicit references to other texts.

The book deliberately avoids extended motivational discussions and historical digressions, in favor of maintaining focus and brevity. That said, some brief connections between topics are provided to help orient the reader. Certainly, this book does not aim to be a complete reference on manifolds similarly to Michael Spivak's five-volume series on manifolds. Its main mission is to teach the essentials needed for working with manifolds and vector bundles.

I hope this approach offers a clear and inviting path into a beautiful and profound area of mathematics.

Looking ahead, I plan to write a direct sequel to this book covering topics in algebraic and differential topology. The forthcoming volume will explore singular homology, cellular homology, sheaf cohomology, Morse homology, and their connections to the de Rham cohomology developed here. It will also introduce characteristic classes, more specifically Chern classes, along with other advanced topics that mark the transition from classical theory to the frontiers of current research. In short, this volume is intended for first-year graduate students, while the next is aimed at second-year students looking to deepen their understanding of manifolds and related ideas.

Mohammad F. Tehrani

# Acknowledgement

I first learned about manifolds as an undergraduate at Sharif University, in a course taught by the beloved Professor Siavash Shahshahani. His teaching had a profound impact on my appreciation for the subject, and his own book on manifolds [**Sha16**] has significantly influenced this text. In fact, I used his book as the main reference while teaching manifolds at the University of Iowa, and many of the exercises included here are drawn from his extensive collection. I also had the privilege of working with Aleksey Zinger for many years. He taught me how to think clearly and write precisely. In many ways, my writing style is an attempt to emulate his clarity and rigor, while adding my own personal touches. Aleksey has produced detailed notes on a range of topics, including manifolds and vector bundles, and some of the exercises and ideas in this book are inspired by his work.

Among other resources, I have made extensive use of the excellent books by Bott and Tu [**BT82**], and by Griffiths and Harris [**GH94**], in the courses I have taught at Iowa. The sequel to this book will be a humble attempt to blend the content of Bott and Tu's Differential Forms in Algebraic Topology with Hatcher's Algebraic Topology [**Hat02**], offering a unified tour of homology and cohomology theories with particular attention to topics such as Chern classes.

I am deeply grateful to my teachers, to the colleagues from whom I have learned mathematics, to the authors of the outstanding references that have made the subject more accessible, and to the students who have patiently and enthusiastically explored these ideas with me in class.

Chapter 1

# Continuous manifolds

Roughly speaking, a **manifold** is a topological space that locally resembles Euclidean space. Globally, a manifold is constructed by patching together countably many such local pieces, called **charts**. In general, manifolds are not homeomorphic to Euclidean space or even to an open subset of it. For example, the sphere is not homeomorphic to the plane. In the following sections, we will introduce tools for distinguishing between different manifolds.

Manifolds appear in many areas of mathematics and physics and are often equipped with additional structures – such as a metric, a holomorphic structure, or a symplectic form – depending on the context. Here, we will primarily focus on manifolds endowed with either a **differentiable** or a **holomorphic** structure. Although there exist topological manifolds that do not admit any smooth structure, the category of smooth manifolds includes nearly all classical and well-studied examples. A differentiable structure enables the generalization of calculus on Euclidean space to manifolds.

We begin by recalling some definitions and results from general topology; for details, see [**Mun75**].

**Definition 1.1.** Let $M$ be a topological space. We say that $M$ is:

(1) *Hausdorff*, if any two distinct points in $M$ can be separated by disjoint open sets;

(2) *Regular*, if singleton sets are closed[1], and for every point $p \in M$ and any closed subset $C \subset M$ not containing $p$, there exist disjoint open sets separating $p$ and $C$;

[1] Alternatively, one may assume $M$ is Hausdorff.

(3) *Normal*, if singleton sets are closed, and any two disjoint closed subsets of $M$ can be separated by disjoint open sets;

(4) *Paracompact*, if every open cover of $M$ admits a refinement that is locally finite—that is, each point in $M$ has a neighborhood intersecting only finitely many sets in the refinement;

(5) *Metrizable*, if the topology on $M$ is induced by a metric (i.e. a distance function);

(6) *Second-countable*, if $M$ has a countable basis. That is, there exists a collection of open sets $\mathcal{B} = \{U_\alpha\}_{\alpha \in \mathcal{I}}$ such that every open set in $M$ can be written as a union of sets in $\mathcal{B}$, and $\mathcal{I}$ is a countable index set.

The following results relate some of these properties. First, note that second-countability is a stronger condition than metrizability. Moreover, every metrizable space is normal, and hence also regular and Hausdorff. The next theorem also shows that every metrizable space is paracompact.

**Theorem 1.2** (Urysohn Metrization Theorem ([**Mun75**], Theorem 34.1)). *Every regular and second-countable topological space is metrizable.*

**Theorem 1.3** (Smirnov Metrization Theorem ([**Mun75**], Theorem 42.1)). *A topological space $M$ is metrizable if and only if it is Hausdorff, paracompact, and locally-metrizable.*

The following diagram provides a rough summary of the relationships among these topological properties. An arrow from one property to another indicates that the former implies the latter.

$$\begin{tikzcd}
\text{second countable} \arrow[r, "+\text{ regular}"] & \text{metrizable} \arrow[r] \arrow[d] & \text{paracompact} & \\
& \text{normal} \arrow[r] & \text{regular} \arrow[r] & \text{Hausdorff}
\end{tikzcd}$$

We are now ready to define a $C^0$ (or topological) manifold.

**Definition 1.4.** A **topological manifold** or $C^0$ **manifold** $M$ is a topological space that is both Hausdorff and second-countable, and satisfies the following condition: for every point $p \in M$, there exists an open neighborhood $U \ni p$ and a homeomorphism $\varphi\colon U \longrightarrow V$ onto an open subset $V \subset A$, where $A$ is a finite-dimensional real or complex vector space[2].

A local homeomorphism $\varphi\colon U \to V$ as in Definition 1.4 is called a (local) **chart** for $M$ around the point $p$. If $A = \mathbb{R}^m$ or $A = \mathbb{C}^m$ and

$$\varphi(p) = \big(x_1(p), \ldots, x_m(p)\big) \qquad \forall\ p \in U,$$

[2]The letter $A$ stands for **affine** space.

then the functions $(x_i)_{i=1}^m$ are called the (real or complex) **local coordinates** around $p$ associated with the chart $\varphi$; see Figure 1.

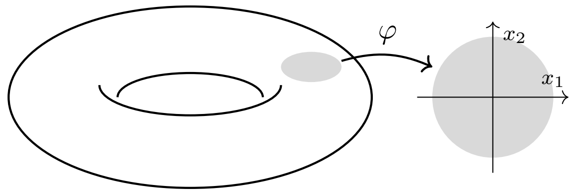


**Figure 1.** A chart on a torus.

**Remark 1.5.** (1) In many standard definitions of manifolds, one often sees $A = \mathbb{R}^m$ rather than an abstract vector space. However, this restriction is not always necessary. The more general form of Definition 1.4 allows for both real and complex charts. While every finite-dimensional real (respectively, complex) vector space is isomorphic to $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$), such an identification depends on a choice of basis. In many contexts, especially those lacking a preferred basis, the abstract formulation is more natural. Since linear maps between vector spaces are smooth, this generality introduces no complications when we later define smooth manifolds.

(2) Every chart map $\varphi$ is a homeomorphism, and thus its inverse

$$\varphi^{-1} \colon V \longrightarrow U$$

carries the same amount of information. As a result, one can equivalently define a chart as a homeomorphism from an open subset of an affine space to an open subset of the manifold. In some situations, this latter perspective is more convenient. Throughout, we will alternate freely between the two conventions and refer to both $\varphi \colon U \to V$ and $\varphi^{-1} \colon V \to U$ as charts on $M$, without further comment. We will adopt a similar approach when discussing local trivializations of vector bundles in later sections.

The half-space

$$\mathbb{H}_m = \big\{(x_1, x_2, \dots, x_m) \colon x_1 \geq 0\big\}$$

is not a manifold in the sense of Definition 1.4 along its *boundary points*:

$$\partial\mathbb{H}_m = \{0\} \times \mathbb{R}^{m-1} \subset \mathbb{R}^m.$$

A simple modification of Definition 1.4 allows us to define a manifold $M$ *with boundary* $\partial M$. We will mostly encounter manifolds with boundary when discussing Stokes' Theorem.

**Definition 1.6.** A topological manifold (or $C^0$ manifold) $M$, **possibly with boundary**, is a topological space that is Hausdorff and second countable, and satisfies the following condition: for every point $p \in M$, there exists an open neighborhood $U \ni p$ and a homeomorphism

$$\varphi \colon U \longrightarrow V$$

onto an open subset $V \subset \mathbb{H}_m$.

Figure 2 illustrates a chat map around a boundary point.

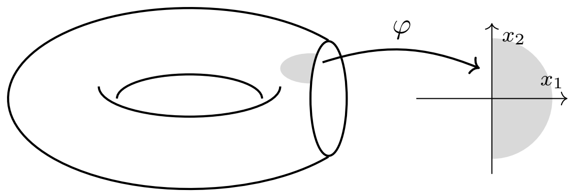


**Figure 2.** A boundary chart on a torus with boundary

Note that Definition 1.6 includes Definition 1.4 as a special case. Charts whose image lies in $\mathbb{H}_m \setminus \partial\mathbb{H}_m$ behave as before.

For all $p \in M$, the condition $\varphi(p) \in \partial\mathbb{H}_m$ is independent of the choice of chart $\varphi$. We call the set of such points the **boundary** of $M$ and denote it by $\partial M$. The set $\partial M$ is itself a topological $(m-1)$-manifold without boundary. In the upcoming statements, when we talk of a neighborhood of a boundary point that is homeomorphic to a ball, we mean an open subset of the form

$$\varphi(U) = B_r(0) \cap \mathbb{H}_m$$

for some $r > 0$. By gluing two copies of $M$ along $\partial M$ via the identity map on $\partial M$, we obtain a manifold without boundary. This construction can be used to reduce certain statements about manifolds with boundary to the case of manifolds without boundary.

It is natural to ask whether the integer $m = \dim_{\mathbb{R}} A$ in Definition 1.4 (or Definition 1.6) can vary from chart to chart. Fortunately, the following theorem of Brouwer shows that this is not the case: the integer $m$ is a topological invariant of any connected[3] $C^0$-manifold $M$. We refer to this integer $m$ as the (real) dimension of $M$, and say that $M$ is an $m$-manifold. Complex dimension of complex manifolds will be defined as half of its real dimension.

[3]Thanks to the existence of local charts, every manifold is locally path-connected. As a result, the notions of connectedness and path-connectedness coincide for manifolds; see [**Mun75**, Theorem 25.5].

**Theorem 1.7** (Brouwer's Invariance of Domain Theorem [**Bro12**]). *Let $V \subset \mathbb{R}^n$ be open, and let $f : V \longrightarrow \mathbb{R}^n$ be an injective continuous map. Then $f(V)$ is open in $\mathbb{R}^n$ and $f$ is a homeomorphism between $V$ and $f(V)$.*

**Exercise 1.8.** Use this theorem to show that $\dim_{\mathbb{R}} A$ is independent of the particular choice of chart on a connected manifold $M$. In other words, charts on a connected $C^0$-manifold must have model spaces of the same (real) dimension.

The three conditions in Definition 1.4 are logically independent: there exist examples of topological spaces that satisfy exactly two of the three conditions, but not all three.

**Example 1.9.** (double origin line) We construct a topological space that is second-countable and admits local charts, but is not Hausdorff. Let

$$M := \frac{\mathbb{R} \times \{\pm\}}{(x,+) \sim (x,-) \qquad \forall x \in \mathbb{R}-\{0\}}$$

with the quotient topology. In other words, $M$ is the topological space obtained by identifying two copies of $\mathbb{R}$ along $\mathbb{R}-\{0\}$ via the identity map. It has two "zero points", denoted $0_+$ and $0_-$, which are the images of $(0,+)$ and $(0,-)$ in the quotient space, respectively. Such non-Hausdorff spaces are generally unsuitable for calculus.

**Example 1.10.** (long line) The so-called long line is a classical example of a non-second-countable that is Hausdorff and admits local charts. It is constructed by "stacking" uncountably many copies of the interval $[0,1)$ in a well-ordered sequence indexed by the first uncountable ordinal $\omega_1$ Formally, the long line is the topological space obtained by taking the ordered set $[0,1) \times \omega_1$ with the lexicographic order and equipping it with the order topology. The resulting space is locally homeomorphic to $\mathbb{R}$, Hausdorff, and connected, but it is not second-countable, and hence not a manifold in our sense.

The second-countability condition has many important consequences. To begin with, it follows from the Urysohn Metrization Theorem that every manifold is metrizable. As a result, manifolds possess all the desirable topological properties listed in Definition 1.1. Additionally, many constructions on manifolds proceed in two steps. First, one defines local objects – such as functions, vector fields, or differential forms – on individual charts, where one can apply standard tools from calculus on open subsets of $\mathbb{R}^m$. Second, one assembles these local pieces into a global structure on $M$. For this second step to work, it is crucial that we can form either a locally finite or countable (and hence convergent) sum. The existence of a countable basis – and equivalently, the paracompactness of $M$ – ensures that such constructions are feasible. In particular, every manifold admits (c.f. [**Mun75**, Theorem 41.7]) a partition of unity in the following general topological sense.

*For any subset $Y \subset M$, we denote the closure of $Y$ in $M$ by $\mathrm{cl}_M(Y)$.*

**Definition 1.11.** Suppose $M$ is a topological space and $\mathcal{U} := \{U_\alpha\}_{\alpha\in\mathcal{I}}$ is an open covering of $M$. A **partition of unity** subordinate to $\mathcal{U}$ is a collection of continuous functions

$$\big\{\varrho_\alpha\colon U_\alpha \longrightarrow [0,1]\big\}_{\alpha\in\mathcal{I}}$$

satisfying the following properties:

(1) The support of each function, defined by

$$\mathrm{supp}(\varrho_\alpha) = \mathrm{cl}_M\Big(\big\{x\in U_\alpha\colon \varrho_\alpha(x)\neq 0\big\}\Big)$$

is contained in $U_\alpha$;

(2) The collection of closed sets $\big\{\mathrm{supp}(\varrho_\alpha)\big\}_{\alpha\in\mathcal{I}}$ is locally-finite[4];

(3) The functions sum to one:

$$\sum_{\alpha\in\mathcal{I}} \varrho_\alpha \equiv 1.$$

Note that the point-wise sum in (3) is well-defined at every point $x \in M$ due to the local finiteness in (2).

**Definition 1.12.** A collection of charts

$$\mathcal{A} = \big\{\varphi_\alpha\colon U_\alpha \longrightarrow V_\alpha\big\}_{\alpha\in\mathcal{I}}$$

on a topological space is called an **atlas** if the domains $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ form an open cover of $M$.

Combining two of the defining properties of a manifold, we obtain the following equivalent characterization of a continuous manifold.

**Lemma 1.13.** *A topological space $M$ is a manifold if and only if it is Hausdorff and admits a countable atlas.*

**Proof.** Suppose $M$ admits a countable atlas

$$\mathcal{A} = \big\{\varphi_n\colon U_n \longrightarrow V_n\big\}_{n=1}^{\infty}.$$

Each $U_n$ is second countable because it is homeomorphic to an open subset of an affine space (or half of it). A countable union of second countable spaces is second countable. Therefore, $M$ is second countable.

[4] That is, for every $x \in M$, there exists a neighborhood $U \ni x$ such that $U \cap \mathrm{supp}(\varrho_\alpha) \neq \emptyset$ for only finitely many $\alpha$.

Conversely, suppose $M$ has a countable basis

$$\mathcal{B} = \{U_n\}_{n=1}^{\infty}$$

and can be covered by charts. Let $\mathcal{B}' \subset \mathcal{B}$ be the subcollection of those open sets $U_n$ that are the domain of some chart map $\varphi_n \colon U_n \to V_n$ (and fix one such $\varphi_n$ for every such $U_n$). We show that

$$\mathcal{A} = \left\{\varphi_n \colon U_n \to V_n \colon U_n \in \mathcal{B}'\right\}$$

is an atlas. For every $p \in M$, take some chart $\varphi \colon U \to V$ such that $p \in U$. By the definition of basis, there is $n$ such that $p \in U_n \subset U$. Therefore, $\varphi|_{U_n} \colon U_n \to \varphi(U_n)$ is a chart. We conclude that $U_n \in \mathcal{B}'$ and

$$M = \bigcup_{U_n \in \mathcal{B}'} U_n.$$

□

**Exercise 1.14.** Modifying Example 1.9, let

$$M := \frac{\mathbb{R} \times \{\pm\}}{(x,+) \sim (x^{-1},-) \qquad \forall x \in \mathbb{R}-\{0\}}$$

with the quotient topology. Show that $M$ is Hausdorff. Therefore, it is a manifold (covered by two charts). Write a continuous map $f \colon M \longrightarrow \mathbb{R}^2$ which is a homeomorphism onto $S^1 \subset \mathbb{R}^2$ (with the subspace topology).

**Exercise 1.15.** Let $M = \mathbb{R}/\mathbb{Z}$, where $\mathbb{Z}$ acts by translation,

$$n \cdot x = x + n, \qquad \forall\, x \in \mathbb{R},\ n \in \mathbb{Z}.$$

(a) Show that $M$ is a topological manifold.
(b) Define an atlas on $M$ using the natural projection $\mathbb{R} \longrightarrow M$.
(c) Write a continuous map $f \colon M \longrightarrow \mathbb{R}^2$ which is a homeomorphism onto $S^1 \subset \mathbb{R}^2$.

**Exercise 1.16.** Prove that every connected 1-dimensional manifold $M$ is homeomorphic to $\mathbb{R}$ or $S^1$.

We conclude with a theorem that will be used in Section 5 to show that the universal cover of any manifold is also a manifold.

**Theorem 1.17.** *The fundamental group of any connected manifold $M$ is countable.*

**Proof.** Fix a countable atlas

$$\mathcal{A} = \left\{\varphi_n \colon U_n \longrightarrow V_n\right\}_{n=1}^{\infty}$$

on $M$ such that each $V_n$ is an open ball (in particular, simply connected). For any $n, n'$, the intersection $U_n \cap U_{n'}$ has at most countably many connected components (since each component contains a point with rational coordinates).

Let $S$ be a set containing one point from each such connected component for all $n, n'$. For every pair $x, y \in S$ and $n$ such that $x, y \in U_n$, fix a path from $x$ to $y$ lying entirely in $U_n$. Let $E$ be the (countable) set of all such paths.

Choose a base point $x_0 \in S$. Any loop in $M$ based at $x_0$ is homotopic to a finite concatenation of paths from $E$. Since the set of finite sequences from a countable set is itself countable, it follows that $\pi_1(M)$ is at most countable. $\square$

## Solutions to exercises

**Exercise 1.8.** Suppose $\varphi\colon U \to V \subset \mathbb{R}^n$ and $\varphi'\colon U' \to V' \subset \mathbb{R}^m$ are two overlapping charts (i.e., $U \cap U' \neq \emptyset$) on a topological manifold $M$, with $m \leq n$. Consider the map

$$\iota \circ \varphi' \circ \varphi^{-1}\colon \varphi(U \cap U') \longrightarrow \iota \circ \varphi'(U \cap U') \subset \mathbb{R}^n,$$

where $\iota\colon \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ is the standard inclusion. By Brouwer's Invariance of Domain Theorem, this map is a homeomorphism onto its image. This implies that $m = n$.

Since $M$ is connected, any two charts can be joined by a finite chain of overlapping charts. Therefore, all charts on $M$ map into affine spaces of the same fixed real dimension. □

**Exercise 1.14.** For $x \in \mathbb{R}$, let $x_+$ and $x_-$ denote the images of $(x, +)$ and $(x, -)$ in the quotient space $M$, respectively. Note that

$$x_+ = (1/x)_- \qquad \text{for all } x \neq 0.$$

Let $\pi\colon \mathbb{R} \times \{\pm\} \to M$ denote the quotient projection. We leave it to the reader to verify that $\pi$ is an open map.

To show that $M$ is Hausdorff:

- For any $\epsilon < 1$, the points $0_+$ and $0_-$ can be separated by the disjoint open sets
$$\pi\big((-\epsilon, \epsilon) \times \{+\}\big) \quad \text{and} \quad \pi\big((-\epsilon, \epsilon) \times \{-\}\big).$$
- For any $x \neq 0$ and $\epsilon < |x|/2$, the points $0_\pm$ and $x_\pm$ can be separated by disjoint open sets
$$\pi\big((-\epsilon, \epsilon) \times \{\pm\}\big) \quad \text{and} \quad \pi\big((x - \epsilon, x + \epsilon) \times \{\pm\}\big).$$

Because $\pi$ is open, the maps

$$\varphi_+\colon \mathbb{R} \to M, \qquad \varphi_\pm(x) = x_\pm,$$

define charts on $M$, giving a two-chart atlas (using the viewpoint of Remark 1.5.2). By Lemma 1.13, $M$ is a manifold.

There are many ways to define a topological embedding $f\colon M \longrightarrow \mathbb{R}^2$. One particular choice that will be generalized to spheres of all dimension in the next lecture comes from inverse stereographic projections

$$f\colon M \to S^1 \subset \mathbb{R}^2, \qquad x_\pm \to \left(\frac{2x}{x^2+1}, \pm\frac{x^2-1}{x^2+1}\right).$$

Check the following:

- $f(x_+) = f((1/x)_-)$, so the chart-wise defined map respects the equivalence relation and is well-defined globally on $M$;

- $f$ is bijective onto $S^1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$;
- $f$ is continuous and hence a homeomorphism onto its image.

□

**Exercise 1.15.** Similarly to the previous exercise, we leave it to the reader to verify that the quotient projection $\pi\colon \mathbb{R} \to M$ is an open map. For every $p, q \in M$ with $p \neq q$, there exist distinct $x, y \in [0,1)$ such that $\pi(x) = p$ and $\pi(y) = q$. Without loss of generality, assume $x < y$. Let

$$\epsilon = \min\big((y-x)/2,\ (1+x-y)/2\big).$$

Verify that

$$\pi\big((x-\epsilon, x+\epsilon)\big) \quad \text{and} \quad \pi\big((y-\epsilon, y+\epsilon)\big)$$

are disjoint open sets containing $p$ and $q$, respectively. Therefore, $M$ is Hausdorff. For every $x \in \mathbb{R}$ and $\epsilon < 1/2$, the map

$$(x-\epsilon, x+\epsilon) \to M, \qquad y \mapsto \pi(y),$$

is a chart. The collection of these charts covers the entire $M$. Furthermore, by choosing $x, \epsilon \in \mathbb{Q}$ we obtain a countable atlas. We conclude that $M$ is a manifold. The map

$$f\colon M \longrightarrow \mathbb{C} = \mathbb{R}^2, \qquad \pi(x) \mapsto e^{2\pi \mathrm{i} x} \quad \forall\, x \in \mathbb{R},$$

is well-defined and is a homeomorphism onto the unit circle $S^1 \subset \mathbb{C}$. □

**Exercise 1.16.** By the existence of charts and since $(a,b) \cong_{C^0} \mathbb{R}$, every point $p \in M$ has a connected neighborhood $U$ homeomorphic to $\mathbb{R}$. Inclusion defines a partial order on such neighborhoods. By the Axiom of Choice, let $U$ be a maximal such neighborhood and fix a chart map

$$\varphi\colon U \to \mathbb{R}.$$

If $U = M$, then $M \cong \mathbb{R}$. Otherwise, since $M$ is connected, we have $\mathrm{cl}_M(U) \neq U$. Suppose $p \in \mathrm{cl}_M(U) \setminus U$.

Let $\psi\colon U' \to (-1,1)$ be a chart map defined on an open neighborhood $U'$ of $p$ sending $p$ to $0$. Since $p \in \mathrm{cl}_M(U) \setminus U$, for every $\epsilon > 0$, the set $U'_\epsilon = \psi^{-1}((-\epsilon, \epsilon))$ has non-empty intersection with $U$. Suppose

$$\psi(U'_\epsilon \cap U) = \coprod_{i \in \mathcal{I}} I_i,$$

where each $I_i = (a_i, b_i)$ is an interval. If $|\mathcal{I}| > 2$, then there exists an interval, say $I_0$, such that $a_0 > -\epsilon$ and $b_0 < \epsilon$. Therefore,

$$q = \psi^{-1}(a_0),\ q' = \psi^{-1}(b_0) \in U'_\epsilon \subset U'.$$

For sufficiently small $\epsilon$, the image $\varphi(\psi^{-1}(I_0)) \subset \mathbb{R}$ is an interval strictly contained in $\mathbb{R}$. Taking closure and using Hausdorfness, we conclude that one of $q$ or $q'$ belongs to $U$, which contradicts the assumption that $q, q' \notin U \cap U'_\epsilon$. Therefore, we can assume $|\mathcal{I}| = 1$ or $|\mathcal{I}| = 2$. Furthermore, by the argument above, if $|\mathcal{I}| = 1$, then

$$\psi(V_\epsilon \cap U) = (-\epsilon, b) \quad \text{or} \quad (a, \epsilon)$$

for some $b < 0$ or $a > 0$. Similarly, if $|\mathcal{I}| = 2$, then

$$\psi(U'_\epsilon \cap U) = (-\epsilon, b) \cup (a, \epsilon)$$

for some $b < 0$ and $a > 0$. The condition $p \in \mathrm{cl}_M(U) \setminus U$ then forces $a = 0$ and $b = 0$.

- In the first case, i.e., when $|\mathcal{I}| = 1$, it is relatively easy to show that there exists a chart with domain $U \cup U'_\epsilon$, which contradicts the maximality of $U$.
- In the second case, i.e., when $|\mathcal{I}| = 2$, it is relatively easy to show that $U \cup U'_\epsilon \cong S^1$. Since $S^1$ is connected, it must be the entire $M$.

$\square$

Chapter 2

# Spheres and projective spaces

Before we proceed further, we discuss the examples of spheres and real/complex projective spaces, which play an important role in the study of manifolds and more general topological spaces. We will describe explicit atlases for these manifolds and use them in future calculations.

**Spheres.**

For $m \geq 0$, the (unit) $m$-sphere $S^m$ is defined to be the following subspace of $\mathbb{R}^{m+1}$:

$$S^m = \Big\{(x_0, \ldots, x_m)\colon \sum_{i=0}^{m} x_i^2 = 1\Big\}$$

In general, we have the following lemma about subspaces of Euclidean space or any other manifold.

**Definition 2.1.** Suppose $M$ is a $C^0$ manifold (without boundary) and $Y \subset M$ is a subset, with subspace topology. We say $Y$ is **locally graph-like** if for every $p \in Y$, there is an open neighborhood $M \supset U \ni p$ and a chart map $\varphi\colon U \to V_1 \times V_2 \subset A = A' \times A''$ such that $\varphi(Y \cap U)$ is the graph of a continuous map $f\colon V_1 \to V_2$. In other words,

$$\varphi(q) = (\varphi_1(q), \varphi_2(q)) = \big(\varphi_1(q), f(\varphi_1(q))\big) \quad \forall\, q \in Y \cap U. \tag{2.1}$$

**Lemma 2.2.** *Suppose $M$ is a $C^0$ manifold (without boundary) and $Y \subset M$ is a locally graph-like subset with subspace topology. Then $Y$ is a $C^0$ manifold.*

**Proof.** The subspace topology on $Y$ is automatically Hausdorff and second countable. Thus, it remains to construct an atlas. For any $p \in Y$, consider a chart map $\varphi\colon U \to V_1 \times V_2 \subset A = A' \times A''$ as in Definition 2.1. We will show that

$$\varphi_1\colon U_Y := U \cap Y \to V_1 \subset A_1$$

is a homeomorphism onto its image. By Definition 1.4, the map $\varphi\colon U \longrightarrow V_1 \times V_2$ is a homeomorphism. Therefore, $\varphi_1$ is continuous and surjective. Moreover, by equation (2.1), it is also injective. It remains to show that $\varphi_1^{-1}\colon V_1 \to U_Y$ is continuous, i.e., that $\varphi_1$ is an open map. Since $\varphi$ is a homeomorphism, the sets of the form $\varphi^{-1}(B_1 \times B_2)$, with $B_i \subset V_i$ open, form a basis for the topology on $U$. Furthermore,

$$\varphi_1\big(Y \cap \varphi^{-1}(B_1 \times B_2)\big) = f^{-1}(B_2) \subset V_1.$$

Since $f$ is continuous, $f^{-1}(B_2)$ is open in $V_1$. We conclude that $\varphi_1\colon U_Y \to V_1$ is an open map. □

**Corollary 2.3.** *For $m \geq 0$, $S^m$ is a manifold of dimension $m$.*

**Proof.** For each $i \in \{0, \ldots, m\}$, let

$$\widetilde{U}_i^{\pm} = \Big\{(x_0, \ldots, x_m)\colon \sum_{j \neq i} x_j^2 < 1,\ \pm x_i > 0\Big\} \cong B_1^m(0) \times \mathbb{R}_+$$

where $B_1^m(0)$ is the open ball of radius one around the origin in $\mathbb{R}^m$. Since $Y \cap \widetilde{U}_i^{\pm}$ is the graph of

$$x_i = \pm\Big(1 - \sum_{j \neq i} x_j^2\Big)^{1/2},$$

Lemma 2.2 shows that $S^m$ is a manifold (of dimension $m$). Let

$$U_i^{\pm} := \widetilde{U}_i^{\pm} \cap S^m \quad \text{and} \quad V_i = \Big\{(x_j)_{j \in \{0,\ldots,\widehat{i},\ldots,m\}}\colon \sum_{j \neq i} x_j^2 < 1\Big\} \cong B_1^m(0) \subset \mathbb{R}^m.$$

Since the collection $\{\widetilde{U}_i^{\pm}\}$ covers the entire $\mathbb{R}^{m+1}$, we conclude that the collection of projection maps

$$\mathcal{A} = \Big\{\varphi_{i,\pm}\colon U_i^{\pm} \longrightarrow V_i, \quad (x_0, \ldots, x_m) \to (x_j)_{j \neq i}\Big\}_{(i,\pm),\ i=0,\ldots,m} \tag{2.2}$$

defines an atlas on $S^m$. □

Next, generalizing the construction from the solution to Exercise 1.14, we define a different atlas on $S^m$ consisting of only two charts.

Let

$$p_{\pm} = (\pm 1, 0, \ldots, 0) \in S^m.$$

Projection maps $\varphi_{\pm}$ from $p_{\pm}$ onto the hyperplane

$$A = (x_0 = 0) \cong \mathbb{R}^m \subset \mathbb{R}^{m+1}$$

define homeomorphisms

$$\varphi_\pm \colon U_\pm := S^m \setminus \{p_\pm\} \to A, \qquad \varphi_\pm(x_0, x_1, \ldots, x_m) = \frac{1}{1 \mp x_0}(x_1, \ldots, x_m);$$

see Figure 1.

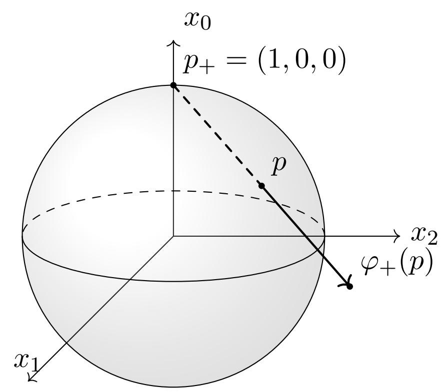


**Figure 1.** Stereographic projection $\varphi_+$ from the north pole $p_+ = (1, 0, 0)$ onto the $xy$-plane

The atlas

$$\mathcal{A} = \{\varphi_\pm \colon U_\pm \to \mathbb{R}^m\} \tag{2.3}$$

provides an efficient covering of the sphere and will be useful in many computations later. It is also important from the perspective of classical geometry, as it gives a concrete realization of $S^m$ as the one-point compactification of $\mathbb{R}^m$.

**Exercise 2.4.** Consider

$$M := \left\{(x, y, z) \in \mathbb{R}^3 \colon x^3 + y^3 + z^3 = 1\right\} \subset \mathbb{R}^3 \tag{2.4}$$

equipped with the subspace topology. Describe an atlas on $M$ to conclude that it is a $C^0$ 2-manifold. Is $M$ connected?

**Real and complex projective spaces.**

**Definition 2.5.** Let $A$ be a vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. The projective space $\mathbb{P}(A)$ is the set of 1-dimensional[1] subspaces (called **lines**) $\ell \subset A$.

[1] When working over $\mathbb{R}$, this means real dimension; when working over $\mathbb{C}$, this means complex dimension.

Note that every line $\ell \subset A$ is of the form $\mathbb{F} \cdot v$ for some vector $v \in A$, and two nonzero vectors $v, v' \in A \setminus \{0\}$ generate the same line if and only if $v' = \lambda v$ for some $\lambda \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$. Therefore,

$$\mathbb{P}(A) = \frac{A \setminus \{0\}}{\mathbb{F}^*}, \tag{2.5}$$

where $\mathbb{F}^*$ acts on $A$ by scalar multiplication. Assuming $A$ is finite-dimensional, the quotient description (2.5) allows us to topologize the set $\mathbb{P}(A)$ using the quotient topology: if

$$\pi \colon A \setminus \{0\} \to \mathbb{P}(A) \tag{2.6}$$

denotes the projection map, then a subset $U \subset \mathbb{P}(A)$ is open if and only if its pre-image $\pi^{-1}(U)$ is open.

**Exercise 2.6.** If $A' \subset A$ is a linear subspace, then $\mathbb{P}(A') \subset \mathbb{P}(A)$ is a closed subset.

In the following, we assume $A$ is finite-dimensional.

**Lemma 2.7.** *If $V \subset A$ is a codimension-one linear subspace, then $U_V := \mathbb{P}(A) \setminus \mathbb{P}(V)$ is an open subset homeomorphic to $V$. Furthermore, any vector $v \in A \setminus V$ determines a homeomorphism (i.e., a chart map)*

$$\varphi_{(V,v)} \colon U_V \to V.$$

**Remark 2.8.** The statement of Lemma 2.7 highlights the value of defining chart maps in Definition 1.4 to take values in an abstract vector space. If $V$ is $n$-dimensional then $V \cong \mathbb{F}^n$; however, these identifications are not canonical and require a choice of basis (as we will do in the calculations below).

**Proof of Lemma 2.7**. By Exercise 2.6, $U_V \subset \mathbb{P}(A)$ is open. The points in $U_V$ correspond to lines $\ell \subset A$ which are not included in $V$. Every such line is of the form

$$\ell = \mathbb{F} \cdot (v \oplus w)$$

for some unique element $w = w(\ell) \in V$; see Figure 2. We leave it to the reader to show that the one-to-one and surjective map

$$\varphi_{(V,v)} \colon U_V \to V, \qquad \ell \longrightarrow w(\ell)$$

is indeed a homeomorphism. Note that $\varphi_{(V,v)}$ maps the line $\mathbb{F} \cdot v$ to $0 \in V$. $\square$

**Exercise 2.9.** For $v, v' \in A \setminus V$, find the relation between

$$\varphi_{(V,v)} \colon U_V \to V \quad \text{and} \quad \varphi_{(V,v')} \colon U_V \to V.$$

**Lemma 2.10.** *For every finite-dimensional vector space $A$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the projective space $\mathbb{P}(A)$ is a manifold of dimension $\dim_{\mathbb{F}} A - 1$. Here, the dimension is understood as real dimension when $\mathbb{F} = \mathbb{R}$ and as complex dimension when $\mathbb{F} = \mathbb{C}$.*

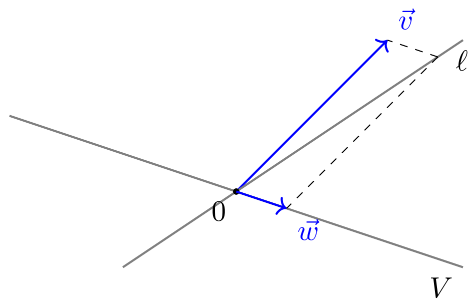


**Figure 2.** Chart maps $\varphi_{(V,v)}$, sending $\ell$ to $w$.

**Proof.** It is clear that the collection

$$\mathcal{A} = \left\{\varphi_{(V,v)} \colon U_V \to V\right\}_{V \subset A,\ \mathrm{codim}_A V = 1,\ v \notin V} \tag{2.7}$$

defines an atlas. Identifying $A \cong \mathbb{F}^n$, let

$$V_i = \{(x_1, \dots, x_n) \in \mathbb{F}^n \colon x_i = 0\},$$

and let $v_i = e_i$ be the $i$-th standard basis vector. Then, it is straightforward to verify that

$$\mathcal{A}_{\mathrm{std}} = \left\{\varphi_i := \varphi_{(V_i, e_i)} \colon U_i := U_{V_i} \to V_i\right\}_{i=1}^{n}$$

is a finite subatlas of (2.10).

It remains to verify that $\mathbb{P}(A)$ is Hausdorff. For any two distinct lines $\ell, \ell' \in \mathbb{P}(A)$, choose a hyperplane $V \subset A$ such that $\ell, \ell' \not\subset V$. Then both $\ell$ and $\ell'$ lie in the open set $U_V$, which is homeomorphic to a Euclidean space. Hence, they can be separated by disjoint open subsets. $\square$

For $A = \mathbb{R}^{n+1}$ and $A = \mathbb{C}^{n+1}$ (i.e., when $A$ is identified with $\mathbb{F}^{n+1}$), the projective spaces $\mathbb{P}(A)$ are denoted by $\mathbb{RP}^n$ and $\mathbb{CP}^n$, respectively. In this case, the equivalence class of a nonzero vector $(X_0, \dots, X_n) \in \mathbb{F}^{n+1}$ in

$$\mathbb{FP}^n = \frac{\mathbb{F}^{n+1} \setminus \{0\}}{\mathbb{F}^*}$$

is written as $[X_0 : \dots : X_n]$. The variables $(X_0, \dots, X_n)$ are called **projective coordinates**, but they are not functions or coordinates on $\mathbb{FP}^n$ in the usual sense. However, statements such as $X_i = 0$ or $X_i \neq 0$ are meaningful.

As explained in the proof of Lemma 2.10, the spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$ can be covered by $(n+1)$ standard charts:

$$\varphi_i \colon U_i \to V_i, \quad \varphi_i\big([X_0 \colon \dots : X_n]\big) = \left(x_j = \frac{X_j}{X_i}\right)_{j \neq i}, \quad \text{for } i = 0, \dots, n, \tag{2.8}$$

where $U_i$ is the open subset defined by $X_i \neq 0$ and $V_i = \mathbb{F}^{\{0, \dots, \widehat{i}, \dots, n\}} \cong \mathbb{F}^n$.

**Exercise 2.11.** Prove that

(1) $\mathbb{RP}^0 = \mathbb{CP}^0 =$ one point.
(2) $\mathbb{RP}^1$ is homeomorphic to $S^1$.
(3) $\mathbb{CP}^1$ is homeomorphic to $S^2$.
(4) $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$ for all $n \geq 2$.
(5) $\mathbb{CP}^n$ is simply-connected for all $n \geq 0$.

**Exercise 2.12.** Given a vector space $A$, let $\mathrm{Gr}_k(A)$ denote the set of $k$-dimensional subspaces of $A$. This is called the **Grassmann manifold**. When $A$ is identified with $\mathbb{R}^n$ or $\mathbb{C}^n$, and the underlying field is clear from context, one usually writes $\mathrm{Gr}(k, n)$ instead of $\mathrm{Gr}_k(A)$. Generalizing the construction of projective space (i.e. $k = 1$), show that $\mathrm{Gr}_k(A)$ is a topological manifold, $\mathrm{Gr}(k, n)$ can be covered by $\binom{n}{k}$ charts, and

$$\dim \mathrm{Gr}(k, n) = k(n - k).$$

## Solutions to exercises

**Exercise 2.4.** Clearly, $M$ is the graph of the continuous function

$$f(x,y) = \sqrt[3]{1 - x^3 - y^3}$$

defined on all of $\mathbb{R}^2$. Therefore, the projection map

$$M \longrightarrow \mathbb{R}^2, \qquad (x,y,z) \mapsto (x,y),$$

serves as a global chart for $M$. We conclude that $M$ is a manifold homeomorphic to $\mathbb{R}^2$, and hence it is connected. $\square$

**Exercise 2.6.** Pre-image of $\mathbb{P}(A) \setminus \mathbb{P}(A')$ in $A \setminus \{0\}$ under the projection map (2.6) is $A - A'$, which is open because $A' \subset A$ is a closed subset. $\square$

**Exercise 2.9.** Suppose $v' = \lambda v \oplus w_0$ for some $w_0 \in V$ and $\lambda \neq 0$. For any $\ell \in U_V$, if

$$\ell = \mathbb{F} \cdot (v' \oplus w), \quad \text{with } w \in V,$$

then

$$\ell = \mathbb{F} \cdot (\lambda v \oplus (w + w_0)) = \mathbb{F} \cdot (v \oplus \lambda^{-1}(w + w_0)).$$

We conclude that

$$\varphi_{(V,v')}(\ell) = \lambda \varphi_{(V,v)}(\ell) - w_0,$$

i.e., $\varphi_{(V,v')}$ is obtained from $\varphi_{(V,v)}$ by composing with a scaling and translation on $V$. $\square$

**Exercise 2.11.** (1) If $A$ is one-dimensional, then $\mathbb{P}(A)$ consists of a single line.

(2) and (4): Every real line in $\mathbb{R}^{n+1}$ is generated by a unit vector $v \in S^n$. Furthermore, two unit vectors $v, v' \in S^n$ generate the same line if and only if $v' = \pm v$. We conclude that

$$\mathbb{RP}^n = \frac{S^n}{\mathbb{Z}_2},$$

which yields the same quotient topology as before. For $n = 1$, the map

$$\frac{S^1}{\mathbb{Z}_2} \longrightarrow S^1, \qquad [\mathrm{e}^{\mathrm{i}\theta}] \mapsto \mathrm{e}^{2\mathrm{i}\theta}$$

is the desired homeomorphism. Here, $[\mathrm{e}^{\mathrm{i}\theta}]$ denotes the equivalence class of $\mathrm{e}^{\mathrm{i}\theta} \in S^1 \subset \mathbb{R}^2 \cong \mathbb{C}$ under the quotient. For $n \geq 2$, since $S^n$ is simply-connected, the projection map

$$S^n \to \mathbb{RP}^n$$

is a covering map with deck transformation group $\mathbb{Z}_2$. We conclude that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$.

(3) To prove that $\mathbb{CP}^1$ is homeomorphic to $S^2$, we compare the standard two-chart covering of $\mathbb{CP}^1$ in (2.8) with the two-chart covering of $S^2$ in (2.3). For $n = 1$, (2.8) shows that $\mathbb{CP}^1$ can be covered by two charts

$$\varphi_i \colon U_i \to V_i = \mathbb{C}, \quad i = 0, 1, \tag{2.9}$$

with the following properties:

- $\varphi_0(U_0 \cap U_1) = \varphi_1(U_0 \cap U_1) = \mathbb{C}^*$;
- the so-called **transition map** $\varphi_1 \circ \varphi_0^{-1} \colon \mathbb{C}^* \to \mathbb{C}^*$ is given by $z \mapsto z^{-1}$.

Similarly, for $n = 1$, (2.3) shows that $S^2$ can be covered by two charts

$$\varphi_\pm \colon U_\pm \to V_\pm = \mathbb{R}^2,$$

with the following properties:

- $\varphi_+(U_+ \cap U_-) = \varphi_-(U_+ \cap U_-) = \mathbb{R}^2 \setminus \{0\}$;
- the transition map $\varphi_- \circ \varphi_+^{-1} \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is
$$(x_1, x_2) \mapsto \frac{1}{x_1^2 + x_2^2}(x_1, x_2).$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$ via $z = x_1 + \mathrm{i}x_2$, we find that $\varphi_- \circ \varphi_+^{-1}(z) = \bar{z}^{-1}$. Define the maps

$$\begin{aligned} f_{+\mapsto 0} \colon U_+ \to U_0, &\qquad f_{+\mapsto 0}(\varphi_+^{-1}(x_1, x_2)) = [1 : x_1 + \mathrm{i}x_2], \\ f_{-\mapsto 1} \colon U_- \to U_1, &\qquad f_{-\mapsto 1}(\varphi_-^{-1}(x_1, x_2)) = [x_1 - \mathrm{i}x_2 : 1]. \end{aligned}$$

The calculations above show that

$$f_{+\mapsto 0}|_{U_+ \cap U_-} = f_{-\mapsto 1}|_{U_+ \cap U_-}.$$

Therefore, these maps patch together to define a well-defined map

$$f \colon S^2 \to \mathbb{CP}^1.$$

It is straightforward to verify that $f$ is a homeomorphism. In Section 4, we will generalize and further explain the idea behind this construction.

(5) The complement of any chart $U_i \cong \mathbb{C}^n$ in (2.8), say $U_0$, is $\mathbb{CP}^{n-1}$. Since $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ has real codimension 2, any loop $\gamma \subset \mathbb{CP}^n$ can be deformed to avoid $\mathbb{CP}^{n-1}$; that is, $\gamma$ is homotopic to a loop entirely contained in $U_0$. As $U_0$ is simply-connected, it follows that $\mathbb{CP}^n$ is simply-connected. $\square$

**Exercise 2.12.** Similarly to Lemma 2.7, if $V \subset A$ is a codimension $k$ linear subspace, then

$$U_V := \{W \in \mathrm{Gr}_k(A) : W \cap V = \{0\}\}$$

is an open subset. Suppose $v_1, \ldots, v_k \in A$ are $k$ linearly independent vectors such that

$$A = V + \langle v_1, \ldots, v_k \rangle .$$

Since $\mathrm{codim}_A V = k$, this is a direct sum decomposition. Such a collection of $k$ vectors determines a homeomorphism (i.e., a chart map)

$$\varphi_{(V,v_1,\ldots,v_k)} \colon U_V \to V^k$$

in the following way. The points in $U_V$ correspond to $k$-dimensional subspace $W \subset A$ satisfying $A = V + W$. Every such $W$ is the graph of a linear map

$$h\colon \langle v_1, \ldots, v_k \rangle \to V.$$

Let $w_i = h(v_i)$ for all $i = 1, \ldots, k$. We leave it to the reader to show that the one-to-one and surjective map

$$\varphi_{(V,v_1,\ldots,v_k)} \colon U_V \to V^k, \qquad W \longrightarrow (w_1, \ldots, w_k)$$

is indeed a homeomorphism.

It is clear that the collection

$$\mathcal{A} = \left\{\varphi_{(V,v_1,\ldots,v_k)} \colon U_V \to V^k\right\} \tag{2.10}$$

defines an atlas on $\mathrm{Gr}_k(A)$.

Identifying $A \cong \mathbb{F}^n$, for every $1 \le i_1 < \ldots < i_k \le n$, let

$$V_{i_1,\ldots,i_k} = \left\{(x_1, \ldots, x_n) \in \mathbb{F}^n \colon x_{i_a} = 0 \ \forall\ a = 1, \ldots, k\right\},$$

and let $v_a = e_{i_a}$ be the $i_a$-th standard basis vector. Then it is straightforward to verify that

(2.11)

$$\mathcal{A}_{\mathrm{std}} = \left\{\varphi_{i_1,\ldots,i_k} := \varphi_{(V_{i_1,\ldots,i_k}, e_{i_1},\ldots,e_{i_k})} \colon U_{i_1,\ldots,i_k} := U_{V_{i_1,\ldots,i_k}} \to V^k_{i_1,\ldots,i_k}\right\}$$

is a subatlas of (2.10) consisting of $\binom{n}{k}$ charts, and

$$\dim \mathrm{Gr}(k,n) = k \dim(V) = k \times (n-k).$$

It remains to verify that $\mathrm{Gr}_k(A)$ is Hausdorff. For any two distinct elements $W, W' \in \mathbb{P}(A)$, there is a codimension $k$ subspace $V$ such that $W \cap V = W' \cap V = \{0\}$. Then both $W$ and $W''$ lie in the open set $U_V$, which is homeomorphic to a Euclidean space. Hence, they can be separated by disjoint open subsets. $\square$.

Chapter 3

# Smooth and holomorphic manifolds

We will not be able to do calculus on $C^0$ manifolds without placing additional structure on them. For instance, extra assumptions are needed to differentiate functions and thereby extend the notion of derivatives from calculus to manifolds. More precisely, suppose $M$ is a topological $m$-manifold and $f\colon M \to \mathbb{R}$ is a continuous function. Fix a chart

$$\varphi\colon U \to V \subset \mathbb{R}^m$$

on $M$. Then the composition

$$f \circ \varphi^{-1}\colon V \to \mathbb{R}$$

is a function defined on an open subset of $\mathbb{R}^m$, where the standard notion of derivatives (e.g., partial derivatives) applies. In particular, we would like to call $f$ a smooth (i.e., $C^\infty$) function if $f \circ \varphi^{-1}$ is smooth. Similarly, if $m = 2k$ and we identify $\mathbb{R}^{2k}$ with $\mathbb{C}^k$, we would like to call $f\colon M \to \mathbb{C}$ holomorphic if $f \circ \varphi^{-1}$ is holomorphic.

However, this notion of smoothness or holomorphicity depends on the choice of chart in the following way. Suppose $\psi\colon U' \to V'$ is another chart such that $U \cap U' \neq \emptyset$. Then, restricted to $\varphi(U \cap U') \subset \mathbb{R}^m$, we have

$$f \circ \varphi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1}),$$

where the **transition map**

$$\psi \circ \varphi^{-1}\colon \varphi(U \cap U') \to \psi(U \cap U')$$

is a homeomorphism between open subsets of $\mathbb{R}^m$. If the transition map $\psi \circ \varphi^{-1}$ is smooth with a smooth inverse, then the smoothness of $f \circ \psi^{-1}$

and $f\circ\varphi^{-1}$ is equivalent on the overlap. Similarly, for $f\colon M\to\mathbb{C}$, if the transition map is holomorphic with a holomorphic inverse, then holomorphicity is preserved across charts. The same conclusion holds for other regularity conditions, such as being $C^k$, analytic, etc.

**Definition 3.1.** Suppose

$$\mathcal{A}=\left\{\varphi_\alpha\colon U_\alpha\to V_\alpha\right\}_{\alpha\in\mathcal{I}}$$

is an atlas for a topological manifold $M$. We say that $\mathcal{A}$ defines a $C^k$, smooth, analytic, or holomorphic structure on $M$ if the **transition maps** (see Figure 1)

$$\begin{aligned}&\varphi_{\alpha\mapsto\beta}:=\varphi_\beta\circ\varphi_\alpha^{-1}|_{V_{\alpha,\beta}}\colon V_{\alpha,\beta}\to V_{\beta,\alpha},\\ &\text{where}\quad V_{\alpha,\beta}:=\varphi_\alpha(U_\alpha\cap U_\beta),\quad\forall\ \alpha,\beta\in\mathcal{I},\end{aligned}\tag{3.1}$$

are $C^k$, smooth, analytic, or holomorphic, respectively.

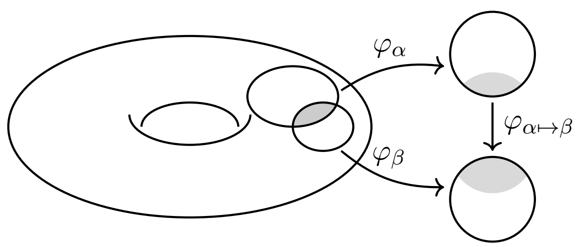


**Figure 1.** Transition maps.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are two smooth atlases on a manifold $M$. If the transition maps between every chart in $\mathcal{A}$ and every chart in $\mathcal{B}$ are smooth, then the union $\mathcal{A}\cup\mathcal{B}$ is a larger atlas that defines the same smooth structure on $M$. In this case, we write $\mathcal{A}\sim\mathcal{B}$. Otherwise, $\mathcal{A}$ and $\mathcal{B}$ define different smooth structures on the same topological manifold $M$. A similar notion applies for $C^k$, analytic, and holomorphic structures on $M$. It is easy to verify that the relation $\sim$ defines an equivalence relation on the set of atlases.

**Definition 3.2.** A **smooth**, $C^k$, **analytic**, or **holomorphic** structure on a topological manifold $M$ is an equivalence class $[\mathcal{A}]$ of atlases such that each atlas in the class defines a smooth, $C^k$, analytic, or holomorphic structure on $M$, respectively.

Every smooth, $C^k$, analytic, or holomorphic structure on $M$ has a unique **maximal atlas** representing it.

**Example 3.3.** It is straightforward to verify that if $M$ and $N$ are $C^k$, smooth, analytic, or holomorphic manifolds, then the product manifold $M\times$

$N$, equipped with the product atlas, is also a manifold of the same regularity. For instance, since $S^1$ is smooth, the $k$-dimensional tori $T^k := (S^1)^k$ are all smooth manifolds as well.

**Remark 3.4.** If $M$ is a smooth manifold with boundary, then $\partial M$ naturally inherits a smooth structure from $M$. The analogous statement does not make sense in the holomorphic category due to dimensional constraints.

**Exercise 3.5.** Show that the atlases on $S^m$ and $\mathbb{RP}^m$ defined in Section 2 determine smooth structures. Show that the atlas defined on $\mathbb{CP}^m$ determines a holomorphic structure on $\mathbb{CP}^m$. Show that the two different atlases on $S^m$ constructed in Section 2 are equivalent, and thus define the same smooth structure.

Having defined smooth and holomorphic manifolds, we can now introduce maps between them that satisfy appropriate regularity conditions. These include smooth maps, holomorphic maps, and other variants depending on the chosen structure.

**Definition 3.6.** Suppose $M$ and $M'$ are smooth (respectively, holomorphic) manifolds, and let $f\colon M \to M'$ be a continuous map. We say that $f$ is **smooth** (respectively, **holomorphic**) if for every pair of charts $\varphi\colon U \to V$ on $M$ and $\varphi'\colon U' \to V'$ on $M'$ from the corresponding maximal atlases, the composition

$$\varphi' \circ f \circ \varphi^{-1}\colon \varphi(f^{-1}(U')) \to V' \tag{3.2}$$

is a smooth (respectively, holomorphic) map between open subsets of two affine spaces.

In particular, we define:

- A **diffeomorphism** as a homeomorphism $f\colon M \to M'$ such that both $f$ and $f^{-1}$ are smooth.
- A **biholomorphism** as a homeomorphism $f\colon M \to M'$ such that both $f$ and $f^{-1}$ are holomorphic.

Note that to check whether a map $f\colon M \to M'$ is smooth (respectively, holomorphic) near a point $p \in M$, it is sufficient to verify the smoothness (respectively, holomorphicity) of the composition in (3.2) for a single chart $\varphi\colon U \to V$ around $p$ and a single chart $\varphi'\colon U' \to V'$ around $f(p)$. If we replace $\varphi\colon U \to V$ with another chart $\widetilde{\varphi}\colon \widetilde{U} \to \widetilde{V}$ around $p$, the corresponding composition becomes

$$\varphi' \circ f \circ \widetilde{\varphi}^{-1} = \left(\varphi' \circ f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \widetilde{\varphi}^{-1}\right).$$

Since the transition map $\varphi \circ \widetilde{\varphi}^{-1}$ is smooth (respectively, holomorphic) by assumption, the smoothness (respectively, holomorphicity) of $\varphi' \circ f \circ \varphi^{-1}$

and $\varphi' \circ f \circ \widetilde{\varphi}^{-1}$ are equivalent. The same argument applies when changing the chart on the target manifold $M'$.

**Exercise 3.7.** Let $f\colon M \longrightarrow N$ be a continuos map between smooth manifolds. Show that $f$ is a smooth map if and only if for every smooth function $h\colon N \longrightarrow \mathbb{R}$, the composition function $h \circ f\colon M \longrightarrow \mathbb{R}$ is also smooth.

**Exercise 3.8.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth and everywhere positive function. Consider the graph $y = f(x)$ of this function in $xy$-plane. By revolving this graph in the $xyz$-space around the $x$-axis, we obtain $M$, a "surface of revolution". Consider the following four charts $(U_y^{\pm}, \varphi_{y,\pm})$, $(U_z^{\pm}, \varphi_{z,\pm})$ for $M$:

$$U_y^{\pm} = M \cap \{\pm y > 0\}, \qquad U_z^{\pm} = M \cap \{\pm z > 0\},$$

with $\varphi_{y,\pm}\colon U_y^{\pm} \longrightarrow V_y^{\pm} \subset \mathbb{R}^2$ defined to be the projection to $xz$-plane and $\varphi_{z,\pm}\colon U_z^{\pm} \longrightarrow V_z^{\pm} \subset \mathbb{R}^2$ defined to be the projection to $xy$-plane. Show that the charts above define a smooth atlas for $M$ such that $M$ is diffeomorphic to a cylinder (i.e. $S^1 \times \mathbb{R}$).

**Exercise 3.9.** For $a, b, c, d \in \mathbb{C}$, with $ad - bc \neq 0$, show that any so called **Möbius** function

$$f(z) = \frac{az+b}{cz+d}\colon \mathbb{C} \setminus \{-d/c\} \longrightarrow \mathbb{C}$$

extends to a holomorphic automorphism (i.e. a biholomorphism from a space to itself) of $\mathbb{CP}^1$.

Having discussed various notions of regularity for an atlas on a topological manifold, it is natural and important to consider the following questions:

(1) Does every topological manifold admit at least one $C^1$ structure?

(2) For $r \geq 1$, given a maximal $C^r$ atlas $\mathcal{A}$ on $M$ and $r < k \leq \infty$, does there exist a subatlas $\mathcal{B} \subset \mathcal{A}$ that defines a $C^k$ structure on $M$? If so, is it unique?

(3) Can a topological manifold admit more than one smooth structure (possibly even infinitely many), up to conjugation by homeomorphisms of $M$?

Before we answer these questions, let us explain the meaning and necessaity of "up to conjugation by homeomorphisms of $M$" in the last question.

Given a smooth atlas $\mathcal{A} = \big\{\varphi_\alpha\colon U_\alpha \to V_\alpha\big\}_{\alpha\in\mathcal{I}}$ on $M$ and a homeomorphism $f\colon M \to M$, we can construct a new atlas by pulling back the charts of $\mathcal{A}$ along $f$:

$$\mathcal{A}' := \big\{\varphi'_\alpha = \varphi_\alpha \circ f\colon f^{-1}(U_\alpha) \to V_\alpha\big\}_{\alpha\in\mathcal{I}}.$$

If $f$ is not differentiable, the atlas $\mathcal{A}'$ will generally not be compatible with $\mathcal{A}$ – that is, the transition maps between charts in $\mathcal{A}$ and those in $\mathcal{A}'$ may fail to be smooth. Nevertheless, the homeomorphism $f$ defines a diffeomorphism from the smooth manifold $(M, \mathcal{A}')$ to the original smooth manifold $(M, \mathcal{A})$ in the sense of Definition 3.6. In this sense, the two smooth structures are considered equivalent.

**Example 3.10.** The map $f\colon \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^n$ for some odd positive integer $n$, is a homeomorphism. Therefore,

$$\mathcal{A} = \{\mathrm{id}\colon \mathbb{R} \to \mathbb{R}\} \quad \text{and} \quad \mathcal{A}' = \{f\colon \mathbb{R} \to \mathbb{R}\}$$

are both single-chart atlases on $\mathbb{R}$ defining smooth structures (since there are no transition maps to consider). While $\mathcal{A} \not\sim \mathcal{A}'$, the map $f\colon (\mathbb{R}, \mathcal{A}') \to (\mathbb{R}, \mathcal{A})$ is a diffeomorphism, identifying the two smooth structures.

The answer to the first question is **no** in dimensions greater than 3, and **yes** in dimensions 1, 2, and 3. We will not delve into the long and rich history of this question here, but we note that in dimensions five and higher, there is a classification of smooth, piecewise-linear, and topological structures due to Kirby and Siebenmann [**KS77**], formulated in terms of obstruction theory and various invariants from algebraic topology. In contrast, the case of dimension 4 is exceptionally intricate. It has been shown that there exist uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$ (this answers question 3 positively), and to this day, a full classification of smooth 4-manifolds remains out of reach.

The answer to the second question is fully positive. Every maximal $C^r$ atlas on a manifold $M$ contains a unique maximal smooth subatlas. Moreover, every maximal smooth atlas includes a unique maximal real-analytic subatlas. This justifies our choice in this book to focus exclusively on smooth and holomorphic manifolds.

We conclude this section with stating the following result on smooth partitions of unity, whose proof is essentially identical to the continuous version.

**Theorem 3.11.** *Given any smooth atlas $\mathcal{A} = \left\{\varphi_\alpha\colon U_\alpha \to V_\alpha\right\}_{\alpha\in\mathcal{I}}$ on a manifold $M$, there exists a smooth partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$.*

**Remark 3.12.** Holomorphic manifolds do not admit holomorphic partitions of unity, since a holomorphic function that vanishes on a nonempty open set must vanish identically. This rigidity is a hallmark of complex-analytic geometry and is one reason why certain results in this book apply only to smooth manifolds.

## Solutions to exercises

**Exercise 3.5.** For the atlas (2.2) on $S^m$, we have the following transition maps:

- Since $U_i^+ \cap U_i^- = \emptyset$ for every $i$, there is no transition map.
- For $i \neq j$ and $\varepsilon_i, \varepsilon_j \in \{\pm\}$, we have

$$\varphi_{i,\varepsilon_i}(U_i^{\varepsilon_i} \cap U_j^{\varepsilon_j}) = V_i \cap \{\varepsilon_j x_j > 0\},$$
$$\varphi_{j,\varepsilon_j} \circ \varphi_{i,\varepsilon_i}\big((x_k)_{k\neq i}\big) = \Big((x_k)_{k\neq i,j}, \varepsilon_i\big(1 - \sum_{k\neq i} x_k^2\big)^{1/2}\Big) \in V_j \cap \{\varepsilon_i x_i > 0\},$$

which is clearly smooth on the open half disk $V_i \cap (x_j > 0)$.

For the two-chart atlas (2.3) on $S^m$, the transition map $\varphi_- \circ \varphi_+^{-1} \colon \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$ and its inverse $\varphi_+ \circ \varphi_-^{-1}$ are both given by the same smooth expression:

$$x = (x_1, \ldots, x_m) \mapsto \frac{1}{|x|^2} x.$$

To show that the atlases (2.3) and (2.2) are equivalent, by symmetry in their definitions, it suffices to check the transition maps between $\varphi_+$ and $\varphi_{i,\pm}$. We compute:

$$\varphi_+ \circ \varphi_{0,+}^{-1} \colon V_0 \setminus \{0\} \to \mathbb{R}^m \setminus \mathrm{cl}_{\mathbb{R}^m}(V_0),$$
$$x = (x_1, \ldots, x_m) \mapsto \frac{1}{1 - \sqrt{1 - |x|^2}}(x_1, \ldots, x_m),$$
$$\varphi_+ \circ \varphi_{0,-}^{-1} \colon V_0 \to V_0, \quad x = (x_1, \ldots, x_m) \mapsto \frac{1}{1 + \sqrt{1 - |x|^2}}(x_1, \ldots, x_m).$$

For $i \neq 0$ and $\varepsilon = \pm$, we have:

$$\varphi_+ \circ \varphi_{i,\varepsilon}^{-1} \colon V_i \to \mathbb{R}^m \cap \{\varepsilon x_i > 0\},$$
$$(x_j)_{j\neq i} \mapsto \frac{1}{1 - x_0}\left((x_k)_{k\neq 0,i},\ \varepsilon\sqrt{1 - \textstyle\sum_{j\neq i} x_j^2}\right),$$
$$\varphi_{i,\varepsilon} \circ \varphi_+^{-1}(x = (x_1, \ldots, x_m)) = \frac{1}{1 + |x|^2}\left(|x|^2 - 1, 2x_1, \ldots, \widehat{2x_i}, \ldots, 2x_m\right).$$

In the first two lines above, the transition maps $\varphi_+ \circ \varphi_{0,\pm}^{-1}$ are scalar multiplications by positive smooth functions, so they and their inverses are smooth. The third and fourth lines show that $\varphi_+ \circ \varphi_{i,\varepsilon}^{-1}$ and their inverses are smooth for all $i \neq 0$. We conclude that (2.3) and (2.2) are equivalent and define the same smooth structure on $S^m$.

For both $\mathbb{RP}^m$ and $\mathbb{CP}^m$, with notation as in (2.8), the transition maps between charts are given as follows. For every $0 \leq i, j \leq m$ with $i \neq j$,

$$\varphi_j \circ \varphi_i^{-1}\big((x_k)_{k\neq i}\big) = (y_k)_{k\neq j}, \quad \text{where} \quad y_k = \begin{cases} x_k/x_j & \text{if } k \neq i, \\ 1/x_j & \text{if } k = i. \end{cases}$$

These maps are clearly smooth when the coordinates are real and holomorphic when the coordinates are complex. □

**Exercise 3.7.** It is straightforward to show that the composition of smooth maps between smooth manifolds is smooth. In particular, if $f\colon M \to N$ is smooth, then for every smooth function $h\colon N \to \mathbb{R}$, the composition $h \circ f\colon M \to \mathbb{R}$ is also smooth.

For the converse, suppose that for every smooth function $h\colon N \to \mathbb{R}$, the composition $h \circ f\colon M \to \mathbb{R}$ is smooth. That is, for every chart $\varphi\colon U \to V \subset \mathbb{R}^m$ (or $\mathbb{H}_m$) in the maximal smooth atlas of $M$, the map $h \circ f \circ \varphi^{-1}\colon V \to \mathbb{R}$ is smooth.

Let $p \in M$, and choose charts $\varphi\colon U \to V \subset \mathbb{R}^m$ (or $\mathbb{H}_m$) around $p$ and $\psi\colon \widetilde{U} \to \widetilde{V} \subset \mathbb{R}^n$ around $f(p)$ such that $f(U) \subset \widetilde{U}$. Write $\psi = (x_1, \ldots, x_n)$, where each $x_i$ is the $i$-th coordinate function of $\psi$.

Since smoothness is a local property, choose a compactly supported smooth function $\varrho\colon \widetilde{U} \to \mathbb{R}$ such that $\varrho \equiv 1$ on a neighborhood of $\psi(f(p))$. For each $i$, define the function $h_i \coloneqq \varrho \cdot x_i$, which is smooth on $\widetilde{U}$ and extends trivially to all of $N$. Moreover, $h_i = x_i$ near $f(p)$.

Therefore, on a sufficiently small neighborhood of $p$, the map $\psi \circ f \circ \varphi^{-1}$ coincides with the smooth map

$$\big(h_1 \circ f \circ \varphi^{-1}, \ldots, h_n \circ f \circ \varphi^{-1}\big).$$

This shows that $\psi \circ f \circ \varphi^{-1}$ is smooth near $\varphi(p)$, and hence $f$ is smooth at $p$. Since $p$ was arbitrary, $f$ is smooth on all of $M$. □

**Exercise 3.8.** Note that $U_y^+ \cap U_y^- = \emptyset$ and $U_z^+ = U_z^- = \emptyset$. For $\varepsilon, \varepsilon' \in \{\pm\}$, the transition maps

$$\varphi_{y,\varepsilon'} \circ \varphi_{z,\varepsilon}^{-1}(x, y) = \big(x, \varepsilon\sqrt{f(x)^2 - y^2}\big),$$
$$\varphi_{z,\varepsilon} \circ \varphi_{y,\varepsilon'}^{-1}(x, z) = \big(x, \varepsilon'\sqrt{f(x)^2 - z^2}\big),$$

are clearly smooth. Therefore, these charts define a smooth atlas for $M$.

It is easy to check that

$$f\colon \mathbb{R} \times S^1 \to M, \qquad \big(x, e^{\mathrm{i}\theta}\big) \mapsto \big(x, f(x)\cos\theta, f(x)\sin\theta\big),$$

is injective and surjective. It remains to show that $f$ and $f^{-1}$ are smooth, where the domain has the product smooth structure.

Consider the 4-chart smooth atlas of $S^1 \subset \mathbb{R}^2$ described in (2.2):

$$\Big\{\varphi_{i,\pm}\colon U_i^{\pm} \to V_i, \quad (x_0, x_1) \mapsto (x_j)_{j\neq i}\Big\}_{(i,\pm),\, i=0,1},$$

given by projecting the upper/lower and left/right halves to the $x_0$- and $x_1$-axes. Taking the product with $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ yields a 4-chart smooth atlas on $\mathbb{R} \times S^1$.

In terms of coordinates $(x, (x_0, x_1))$, the map $f$ is the restriction to $\mathbb{R} \times S^1$ of the smooth automorphism

$$\mathbb{R}^3 \to \mathbb{R}^3, \qquad (x, x_0, x_1) \mapsto \big(x, f(x)x_0, f(x)x_1\big).$$

In later chapters we will see that this implies the restriction $f$ is a diffeomorphism. However, the point of this exercise is to directly verify the condition in Definition 3.6.

We compute:

$$\begin{aligned}
\varphi_{y,\varepsilon'} \circ f \circ (\mathrm{id} \times \varphi_{0,\varepsilon})^{-1}(x, x_1) &= \big(x, f(x)x_1\big), && \forall x \in \mathbb{R},\ -1 < x_1 < 1,\\
\varphi_{z,\varepsilon'} \circ f \circ (\mathrm{id} \times \varphi_{0,\varepsilon})^{-1}(x, x_1) &= \big(x, \varepsilon f(x)\sqrt{1-x_1^2}\big), && \forall x \in \mathbb{R},\ 0 < \varepsilon' x_1 < 1,\\
\varphi_{y,\varepsilon'} \circ f \circ (\mathrm{id} \times \varphi_{1,\varepsilon})^{-1}(x, x_0) &= \big(x, \varepsilon f(x)\sqrt{1-x_0^2}\big), && \forall x \in \mathbb{R},\ 0 < \varepsilon' x_0 < 1,\\
\varphi_{z,\varepsilon'} \circ f \circ (\mathrm{id} \times \varphi_{1,\varepsilon})^{-1}(x, x_0) &= \big(x, f(x)x_0\big), && \forall x \in \mathbb{R},\ -1 < x_0 < 1.
\end{aligned}$$

The other cases are vacuous. Clearly, all these maps are smooth with smooth inverses. □

**Exercise 3.9.** Recall from the solution to Exercise 2.11 that $\mathbb{CP}^1$ can be covered by two charts

$$\varphi_i\colon U_i \to V_i = \mathbb{C}, \quad i = 0, 1, \tag{3.3}$$

with the following properties:

- $\varphi_0(U_0 \cap U_1) = \varphi_1(U_0 \cap U_1) = \mathbb{C}^*$;
- the transition map $\varphi_1 \circ \varphi_0^{-1}\colon \mathbb{C}^* \to \mathbb{C}^*$ is given by $z \mapsto w = z^{-1}$, where $z$ is the coordinate on $V_0$ and $w$ is the coordinate on $V_1$.

This shows that $\mathbb{CP}^1 \cong U_0 \cup \{[0:1]\}$ is a one-point compactification of $U_0 \cong \mathbb{C}$. Since $[0:1]$ corresponds to $z = \frac{1}{0} = \infty$, it is common to write $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, allowing $z$ to take values in $\mathbb{C}$ as well as $\infty$.

Under the chart map $\varphi_0\colon U_0 \to \mathbb{C}$ on both domain and target, the Möbius transformation in Exercise 3.9 is the restriction to $U_0 \setminus \{[c : -d]\}$ of the well-defined function

$$f\colon \mathbb{P}^1 \to \mathbb{P}^1, \qquad [X_0 : X_1] \mapsto [dX_0 + cX_1 : bX_0 + aX_1].$$

In other words, under the chart $\varphi_0 \colon U_0 \to \mathbb{C}$, the function $f$ is defined for $p \in \mathbb{CP}^1 \setminus \{[c : -d], [0 : 1]\}$, and we extend $f$ to the missing points by

$$f\big([c : -d]\big) = [0 : 1] \in \mathbb{CP}^1 \quad \text{and} \quad f\big([0 : 1]\big) = [c : a] \in \mathbb{CP}^1.$$

We must show that the resulting map is a holomorphic map between holomorphic manifolds.

**Remark 3.13.** More generally, any real or complex linear map $f \colon A \to A'$ between real or complex affine spaces induces a well-defined map

$$f \colon \mathbb{P}(A) \to \mathbb{P}(A'),$$

since $f$ maps lines to lines. The same principle applies to Grassmannians. For instance, a Möbius transformation corresponds to an invertible linear map

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2, \qquad \begin{bmatrix} X_1 \\ X_0 \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X_1 \\ X_0 \end{bmatrix}.$$

In the complex case, the claim is that $f \colon \mathbb{P}(A) \to \mathbb{P}(A')$ is a holomorphic map. In particular, if $f \colon A \to A$ is invertible, then $f \colon \mathbb{P}(A) \to \mathbb{P}(A)$ is a holomorphic automorphism. We prove this general claim.

Identifying $A$ with $\mathbb{C}^{n+1}$, with coordinates $(X_0, \ldots, X_n)$, and $A'$ with $\mathbb{C}^{m+1}$, with coordinates $(Y_0, \ldots, Y_m)$, $f$ has the form

$$f \colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{m+1}, \qquad \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_n \end{bmatrix} \mapsto \begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_m \end{bmatrix} = \begin{bmatrix} a_{10} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m0} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_n \end{bmatrix}.$$

With notation as in (2.8), let $\varphi_i^D \colon U_i^D \to V_i^D$ denote the charts on the domain for $i = 0, \ldots, n$, and $\varphi_j^T \colon U_j^T \to V_j^T$ the charts on the target for $j = 0, \ldots, m$. Then, for every $i$ and $j$, we have

$$\varphi_j^T \circ f \circ (\varphi_i^D)^{-1}\big((x_k)_{k \neq i}\big) = \big((y_s)_{s \neq j}\big), \qquad y_s = \frac{a_{si} + \sum_{k \neq i} a_{sk} x_k}{a_{ji} + \sum_{k \neq i} a_{jk} x_k}.$$

The right-hand side is a ratio of two linear functions in the variables $x_k$. Therefore, this expression defines a holomorphic function on an open subset of $V_i^D$. Changing the identifications $A \cong \mathbb{C}^{n+1}$ and $A' \cong \mathbb{C}^{m+1}$ corresponds to composing $f$ with additional linear transformations represented by invertible matrices. Since composition with holomorphic maps preserves holomorphicity, the property of $f \colon \mathbb{P}(A) \to \mathbb{P}(A')$ being holomorphic is independent of the chosen bases used in these calculations. □

Chapter 4

# Manifolds as quilted spaces

Definitions 1.4, 1.6, and 3.1 begin with a given topological space $M$ and require the existence of an atlas whose transition maps satisfy certain regularity conditions.

However, in many situations the space $M$ is not explicitly specified in advance. Instead, we construct it by gluing together countably many affine open sets using transition maps that are homeomorphisms, diffeomorphisms, or biholomorphisms. In this scenario, the resulting quotient space is, by construction, locally modeled on affine charts and second-countable. What remains is to verify that the space is Hausdorff in order to conclude that it is a manifold.

In other words, we may think of $M$ as a quilted quotient space formed by gluing a countable collection of open subsets (or simpler manifolds) via a prescribed class of transition maps. For many applications, this construction is either necessary or significantly more efficient. In particular, this approach eliminates the need to explicitly write chart maps and allows us to work solely with the transition maps.

More precisely, let $\{V_\alpha\}_{\alpha\in\mathcal{I}}$ be a countable collection of open subsets of $\mathbb{R}^m$, $\mathbb{H}_m$, $\mathbb{C}^m$, or some abstract real or complex vector space. For each pair of indices $\alpha, \beta \in \mathcal{I}$, suppose there exist open subsets $V_{\alpha,\beta} \subset V_\alpha$ and $V_{\beta,\alpha} \subset V_\beta$ together with transition maps

$$\begin{tikzcd} V_{\alpha,\beta} \arrow[r, bend left, "\varphi_{\alpha\mapsto\beta}"] & V_{\beta,\alpha} \arrow[l, bend left, "\varphi_{\beta\mapsto\alpha}=\varphi^{-1}_{\alpha\mapsto\beta}"] \end{tikzcd} ,$$

which are, depending on the context, homeomorphisms, diffeomorphisms, or biholomorphisms.

Assume further that:

(1) $V_{\alpha,\alpha} = V_\alpha$ for all $\alpha \in \mathcal{I}$, and $\varphi_{\alpha\mapsto\alpha} = \mathrm{id}_{V_\alpha}$;

(2) For all $\alpha, \beta, \gamma \in \mathcal{I}$, we have

$$V_{\alpha,\beta\gamma} = V_{\alpha,\gamma\beta} := V_{\alpha,\beta} \cap V_{\alpha,\gamma} = V_{\alpha,\beta} \cap \varphi^{-1}_{\alpha\mapsto\beta}(V_{\beta,\gamma});$$

this condition ensures that the domains and targets of both sides of the cocycle condition below match;

(3) On $V_{\alpha,\beta\gamma}$, the transition maps satisfy the (compatibility) **cocycle condition**:

$$\varphi_{\alpha\mapsto\gamma} = \varphi_{\beta\mapsto\gamma} \circ \varphi_{\alpha\mapsto\beta}.$$

Under these assumptions, the identification

$$x \sim y \quad \Leftrightarrow \quad x \in V_{\alpha,\beta},\ y \in V_{\beta,\alpha},\ y = \varphi_{\alpha\mapsto\beta}(x)$$

defines an equivalence relation on the disjoint union topological space $\widetilde{M} = \coprod_{\alpha\in\mathcal{I}} V_\alpha$.

Let

$$M := \widetilde{M}/\sim \tag{4.1}$$

denote the resulting quotient topological space.

**Lemma 4.1.** *With notation as above, the space $M$ is a ($C^0$, smooth, or holomorphic, depending on the type of transition maps) manifold if and only if it is Hausdorff. Conversely, any countable atlas on a manifold presents it as a quotient space of the form* (4.1).

**Proof.** For each $\alpha \in \mathcal{I}$, let $\varphi_\alpha\colon V_\alpha \to M$ denote the composition of the inclusion $V_\alpha \hookrightarrow \widetilde{M}$ with the quotient projection map $\pi\colon \widetilde{M} \to M$. The collection

$$\mathcal{A} = \Big\{\varphi_\alpha\colon V_\alpha \to U_\alpha := \pi(V_\alpha) \subset M\Big\}_{\alpha\in\mathcal{I}} \tag{4.2}$$

is a countable atlas (in the sense of Remark 1.5.2). By Lemma 1.13, $M$ is a manifold if and only if it is Hausdorff. The converse direction is immediate from how the transition maps of an atlas are defined in Definition 3.1.

□

**Example 4.2.** Example 1.9 of the double origin line is a non-Hausdorff instance of this construction, where

$$\mathcal{I} = \{\pm\}, \quad V_+ = V_- = \mathbb{R}, \quad V_{+,-} = V_{-,+} = \mathbb{R}^*, \quad \varphi_{\pm\to\mp} = \mathrm{id}_{\mathbb{R}^*}.$$

In contrast, Exercise 1.14 constructs $S^1$ by gluing the same collection of open sets using the different transition map $\varphi_{\pm\to\mp}(x) = 1/x$.

As we mentioned earlier, we can consider a more general version of (4.1) where each $V_\alpha$ is itself a manifold. In this case, it is also useful to relax the first condition before (4.1) and allow $V_{\alpha,\alpha}$ to be a proper subset of $V_\alpha$. Then $\varphi_{\alpha\mapsto\alpha}$ should be an involution; i.e.,

$$\varphi_{\alpha\mapsto\alpha}\circ\varphi_{\alpha\mapsto\alpha}=\mathrm{id}.$$

For instance, consider the following example with $\mathcal{I}=\{1\}$, where

$$V_1=S^1\times(0,3),\qquad V_{1,1}=S^1\times\big((0,1)\cup(2,3)\big),$$

and

$$\varphi_{1\mapsto1}(p,t)=(p,2+t)\qquad\forall\ (p,t)\in S^1\times(0,1).$$

Then the quotient manifold

$$M=V_1/\sim,\qquad (p,t)\sim\varphi_{1\mapsto1}(p,t),$$

is diffeomorphic to the 2-torus $S^1\times S^1$.

Such constructions are common in surgery theory of manifolds. For example, it is well-known (e.g. see [**Rol90**, Ch. 9]) that any 3-manifold can be obtained from $S^3$ by surgery along a link (i.e., a disjoint union of knots). Such a surgery removes a neighborhood of the link and glues it back differently by modifying the transition map.

If two manifolds $M$ and $M'$ are presented as in (4.1) by

$$M=\coprod_{\alpha\in\mathcal{I}}V_\alpha/\sim,\qquad x\sim y\quad\Leftrightarrow\quad x\in V_{\alpha,\beta},\ y\in V_{\beta,\alpha},\ y=\varphi_{\alpha\mapsto\beta}(x),$$

$$M'=\coprod_{\alpha'\in\mathcal{I}'}V'_{\alpha'}/\sim,\qquad x\sim y\quad\Leftrightarrow\quad x\in V'_{\alpha',\beta'},\ y\in V'_{\beta',\alpha'},\ y=\varphi_{\alpha'\mapsto\beta'}(x),$$

then a map $f\colon M\to M'$ corresponds to a collection of maps

$$f_{\alpha\mapsto\alpha'}\colon V_{\alpha;\alpha'}\subset V_\alpha\longrightarrow V_{\alpha'}\qquad\forall\ \alpha\in\mathcal{I},\ \alpha'\in\mathcal{I}',$$

such that

$$\varphi_{\alpha'\mapsto\beta'}\circ f_{\alpha\mapsto\alpha'}=f_{\beta\mapsto\beta'}\circ\varphi_{\alpha\mapsto\beta}\qquad\forall\ \alpha,\beta\in\mathcal{I},\ \alpha',\beta'\in\mathcal{I}',$$

on $V_{\alpha;\alpha'}\cap V_{\alpha;\beta'}\cap V_{\alpha,\beta}$. Here, $V_{\alpha;\alpha'}$ is the intersection of $V_\alpha$ and $f^{-1}(V_{\alpha'})$ under the canonical chart maps in (4.2).

*Chapter 5*

# Discrete quotients

In general, there are multiple ways to construct more complicated manifolds from basic ones. These include:

(1) Taking products;

(2) Gluing several pieces along overlapping regions, as described in the previous section;

(3) Taking quotients by group actions;

(4) Considering level sets of functions.

In this lecture, we study quotients by discrete group actions.

Suppose $M$ is a (continuous, smooth, or holomorphic) manifold and $G$ is a discrete group (probably finite). By **a (right-) action of** $G$ on $M$ we mean a function

$$\psi\colon M\times G\longrightarrow M, \qquad (x,g)\longrightarrow x\cdot g:=\psi(x,g)\in M$$

such that $\psi(-,g)\colon M\longrightarrow M$ is continuous, smooth, or holomorphic for all $g\in G$, depending on the context, and

$$\begin{aligned}&\psi(x,g_1g_2)=\psi(\psi(x,g_1),g_2) \qquad \forall\ g_1,g_2\in G,\ x\in M,\\ &\psi(x,1)=x \qquad \forall\ x\in M.\end{aligned}$$

In particular, each $\psi_g:=\psi(-,g)\colon M\longrightarrow M$ should be a homeomorphism, diffeomorphism, or biholomorphism, depending on the context.

Let

$$M/\psi \text{ or } M/G:=M/\sim, \qquad x\sim y\Leftrightarrow y=x\cdot g \text{ for some } g\in G,$$

denote the quotient space with the quotient topology.

**Theorem 5.1.** *With notation as above, suppose $G$ is a discrete group that acts freely and properly on $M$ in the following sense:*

- **freely:** *for every point $x \in M$ the stabilizer subgroup $G_x = \{g \in G \colon x \cdot g = x\}$ is the trivial subgroup;*
- **properly:** *for every compact subset $K \subset M$, the subset $G_K = \{g \in G : (K \cdot g) \cap K \neq \emptyset\}$ is finite.*

*Then the smooth manifold structure on $M$ induces a unique $C^0$, smooth, or holomorphic manifold structure on the quotient $M/G$ such that*

- *the quotient projection map $\pi \colon M \to M/G$ is continuous, smooth, or holomorphic, respectively;*
- *and $f \colon M/G \to N$ is continuous, smooth, or holomorphic, if and only if $f \circ \pi$ is continuous, smooth, or holomorphic, respectively.*

**Remark 5.2.** If $G$ is finite, then the action is automatically proper. One only needs to check that it is free. If $M$ is compact, then the action is proper if and only if $G$ is finite.

**Remark 5.3.** If $x$ and $y$ belong to the same orbit of the $G$-action, i.e., $y = x \cdot g$ for some $g \in G$, then the isotropy groups $G_x$ and $G_y$ are conjugate:

$$G_y = g^{-1} G_x g.$$

**Example 5.4.** Recall from the solution to Exercise 2.11 that $\mathbb{RP}^m = S^m/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by $x \mapsto -x$ on $S^m \subset \mathbb{R}^{m+1}$.

**Example 5.5.** The action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by translations,

$$\mathbb{R}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{R}^2, \qquad (x, y) \times (m, n) \mapsto (x + m, y + n),$$

is smooth, free, and proper. Taking the product of the space in Example 1.15 with itself, we see that the quotient manifold $\mathbb{R}^2/\mathbb{Z}^2$ is the 2-dimensional torus $S^1 \times S^1$.

In the holomorphic category, there are many holomorphically non-equivalent ways to define an action of $\mathbb{Z}^2$ on $\mathbb{C} \cong \mathbb{R}^2$. For each $\tau$ in the **upper half-plane**

$$\mathcal{H} = \{\tau \in \mathbb{C} \colon \operatorname{Im}(\tau) > 0\}, \tag{5.1}$$

define an action of $\mathbb{Z}^2$ on $\mathbb{C}$ by

$$\psi^\tau \colon \mathbb{C} \times \mathbb{Z}^2 \longrightarrow \mathbb{C}, \qquad z \times (m, n) \mapsto z + m + n\tau.$$

Then, $\mathbb{T}^2_\tau := \mathbb{C}/\psi^\tau$ is a 2-torus with a holomorphic structure (called an **elliptic curve**). Since $\mathbb{R}^2$ is the universal covering space of $\mathbb{T}^2$, every elliptic curve is of the form $\mathbb{T}^2_\tau$ for some $\tau \in \mathcal{H}$. The following exercise characterizes holomorphic structures on $\mathbb{T}^2$ up to isomorphism.

**Exercise 5.6.** With notation as in Example 5.5, show that $\mathbb{T}^2_\tau$ is biholomorphic to $\mathbb{T}^2_{\tau'}$ if and only if

$$\tau' = A \cdot \tau := \frac{a\tau + b}{c\tau + d} \tag{5.2}$$

for some

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}) = \big\{A \in M_{2\times 2}(\mathbb{Z})\colon\ \det(A) = 1\big\}.$$

**Exercise 5.7.** In Exercise 5.6, the formula (5.2) defines a left action of the discrete group $\mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{H}$. This action clearly descends to an action of $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$. Find the isotropy groups of various values of $\tau$ to show that the quotient is not naturally a manifold. This quotient space parametrizes elliptic curves.

**Exercise 5.8.** Reducing the group in Exercise 5.7, let

$$\Gamma_2 \subset \mathrm{PSL}(2,\mathbb{Z}), \qquad \Gamma_2 = \{A \in \mathrm{PSL}(2,\mathbb{Z})\colon A \equiv I_2 \mod 2\}\,.$$

Show that the action of $\Gamma_2$ on $\mathcal{H}$ is free. (The quotient $\mathcal{H}/\Gamma_2$ is a sphere with three punctures.)

**Exercise 5.9.** Let $p$ and $q$ be coprime integers and consider the action of $\mathbb{Z}_p$ on $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ by diffeomorphisms:

$$(z_1, z_2) \mapsto \Big(\mathrm{e}^{2\pi i/p} z_1,\, \mathrm{e}^{2\pi i q/p} z_2\Big)\,.$$

Show that the action is free to conclude that the quotient space $L(p,q) := S^3/\mathbb{Z}_p$ is a 3-dimensional smooth manifold (called the **lens space**). What is the fundamental group of $L(p,q)$?

**Exercise 5.10.** Give an example of a free action that is not proper.

**Proof of Theorem 5.1.** It is easy to see that the quotient projection map $\pi$ is open. For every $p \in M$, choose an open neighborhood $U \ni p$ such that $U$ is the domain of a chart $\varphi\colon U \longrightarrow V$ in the maximal atlas. Pick a smaller open set $U' \subset U$ such that $p \in U'$ and $\mathrm{cl}_M(U')$ is compact in $U$.

Since the action is proper, there are at most finitely many elements in $G$, say $g_1, \ldots, g_k$, such that $\psi_{g_j}(U') \cap U' \neq \emptyset$. Moreover, since $\psi_{g_j}(p) \neq p$ for all $j$ and $M$ is Hausdorff, there exist open neighborhoods $U_j$ of $\psi_{g_j}(p)$ and $U'_j \subset U'$ of $p$ such that

$$U_j \cap U'_j = \emptyset.$$

Let

$$U'' = \bigcap_{j=1}^{k} \psi_{g_j}^{-1}(U_j) \cap \bigcap_{j=1}^{k} U'_j \subset U'.$$

It is straightforward to check that $\psi_g(U'') \cap U'' = \emptyset$ for all $g \in G \setminus \{e\}$ and $p \in U''$. Therefore, every point $p$ admits a chart $\varphi'' := \varphi|_{U''} \colon U'' \to V'' = \varphi(U'')$ such that

$$\pi \colon U'' \to \pi(U'')$$

is a homeomorphism. Thus,

$$\varphi'' \circ \pi^{-1} \colon \pi(U'') \to V''$$

is a chart around $\pi(p) \in M/G$. The set of such charts defines the induced atlas on $M/G$.

That $M/G$ is Hausdorff and second countable follows from the corresponding properties of $M$, together with the fact established above that $\pi$ is locally a homeomorphism.

The transition maps between charts on $M/G$ are compositions of the maps $\psi_g$ with the transition maps of charts on $M$. Therefore, if $M$ is smooth and $G$ acts by diffeomorphisms, then the transition maps of the induced atlas on $M/G$ are smooth as well. Similarly, if $M$ is holomorphic and $G$ acts by biholomorphisms, then the transition maps of the induced atlas on $M/G$ are holomorphic.

The two bullet-point properties follow directly from our definition of the induced atlas on $M/G$. □

Every manifold $M$ is the quotient of its universal cover $\widetilde{M}$ by the group of deck transformations. The following result can be viewed as a sort of converse to Theorem 5.1 in this specific setting.

**Theorem 5.11.** *The universal cover $\widetilde{M}$ of any continuous, smooth, or holomorphic connected manifold is itself a continuous, smooth, or holomorphic manifold, respectively, such that the covering map $\pi \colon \widetilde{M} \longrightarrow M$ is the quotient projection with respect to the action of $\pi_1(M)$ as the group of deck transformations.*

**Proof.** By Theorem 1.17, the fundamental group $\pi_1(M)$ is countable. Recall that, topologically, $\widetilde{M}$ is the space of paths $\gamma$ from a fixed base point $p_0 \in M$ to any point $p \in M$, considered up to homotopy. That is, a point in $\widetilde{M}$ corresponds to the homotopy class $[\gamma]$ of such a path, and the covering map is defined by $\pi([\gamma]) = p$.

Let

$$\mathcal{A} = \big\{\varphi_n \colon U_n \longrightarrow V_n\big\}_{n\in\mathbb{N}}$$

be a countable atlas on $M$ such that each $V_n$ is a ball (and hence simply connected). For each $n$, choose a point $p_n \in U_n$ and fix a preimage $\widetilde{p}_n \in \widetilde{M}$ such that $\pi(\widetilde{p}_n) = p_n$.

Then, $\pi^{-1}(U_n)$ consists of open sets $\{U_{n,\alpha}\}_{\alpha\in\pi_1(M)}$ satisfying:

- $U_{n,1}$ is the component containing $\widetilde{p}_n$,
- the restriction $\pi\colon U_{n,\alpha} \longrightarrow U_n$ is a homeomorphism for each $\alpha$,
- $U_{n,\alpha} = U_{n,1}\cdot\alpha$ under the right-action $\psi_\alpha\colon \widetilde{M} \longrightarrow \widetilde{M}$ of $\alpha$,
- and $U_{n,\alpha}\cap U_{n,\alpha'} = \emptyset$ for $\alpha \neq \alpha'$.

Therefore,

$$\widetilde{\mathcal{A}} = \big\{\varphi_n \circ \pi\colon U_{n,\alpha} \longrightarrow V_n\big\}_{n\in\mathbb{N},\ \alpha\in\pi_1(M)}$$

is a countable atlas on (the Hausdorff space) $\widetilde{M}$. It is straightforward to check that the transition maps of $\widetilde{\mathcal{A}}$ coincide with those of $\mathcal{A}$, and thus inherit the same level of regularity (i.e., continuous, smooth, or holomorphic). □

**Remark 5.12.** For an arbitrary discrete group action as in Theorem 5.1, the fundamental groups of $M$ and $M/G$ are related by a short exact sequence of groups:

$$1 \longrightarrow \pi_1(M) \longrightarrow \pi_1(M/G) \longrightarrow G \longrightarrow 1\ .$$

In this case, the universal cover of $M/G$ is the same as the universal cover of $M$, and the group $G$ can be identified with a quotient of $\pi_1(M/G)$ by the normal subgroup $\pi_1(M)$.

## Solutions to exercises

**Exercise 5.6.** Suppose

$$f: \mathbb{T}^2_{\tau'} \longrightarrow \mathbb{T}^2_\tau$$

is a biholomorphism. Since $\mathbb{C}$ is the universal cover of both complex tori, there is a canonical biholomorphic lift

$$\widetilde{f}: \mathbb{C} \longrightarrow \mathbb{C}$$

of $f$ satisfying $\widetilde{f}(0) = 0$ (recall that such a lift is unique once we specify a pre-image of a base point). The map $f$ induces a group isomorphism $\pi_1(f): \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ between the fundamental groups. Suppose

$$f(\tau') = a\tau + b, \qquad f(1) = c\tau + d$$

for some $a, b, c, d \in \mathbb{Z}$. Then,

$$f(m\tau'+n) = m(a\tau+b)+n(c\tau+d) = (ma+nc)\tau+(mb+nd) \qquad \forall\, m, n \in \mathbb{Z}.$$

The only holomorphic functions $\widetilde{f}: \mathbb{C} \longrightarrow \mathbb{C}$ with linear growth and $\widetilde{f}(0) = 0$ are linear maps; that is, $\widetilde{f}(z) = \lambda z$ for some $\lambda \in \mathbb{C}^*$. This implies

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

The same argument applies in the reverse direction, implying that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible. Therefore, $\det(A) = \pm 1$. Since both $\tau$ and $\tau'$ lie in the upper half-plane $\mathcal{H}$, we conclude that $\det(A) = +1$. The converse follows by reversing the above steps.

**Remark 5.13.** Since $\mathbb{T}^2_\tau \cong \mathbb{T}^2_{1/\tau}$, it suffices to consider parameters $\tau \in \mathcal{H}$. Note that integer matrices $A$ with $\det(A) = -1$ send $\mathcal{H}$ to its complex conjugate $-\mathcal{H}$.

□

**Exercise 5.7.** Suppose

$$\tau = \frac{a\tau + b}{c\tau + d}$$

for some matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Then $\tau$ is a root of the quadratic polynomial

$$c\tau^2 + (d - a)\tau - b = 0,$$

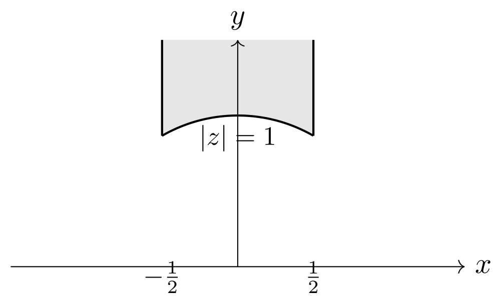


**Figure 1.** Fundamental domain of the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{H}$.

i.e.,

$$\tau = \frac{a-d \pm \sqrt{a^2+d^2-2ad+4bc}}{2c} = \frac{a-d \pm \sqrt{\mathrm{tr}(A)^2-4}}{2c}.$$

Since we care about the class of $A$ in $\mathrm{PSL}(2,\mathbb{Z})$, we may assume $\mathrm{tr}(A) \geq 0$. If $\mathrm{tr}(A) \geq 2$, then $\tau$ is real and does not lie in $\mathcal{H}$. So suppose $a+d<2$. Since $a$ and $d$ are integers, we must have $a+d=0$ or $a+d=1$.

**(1) Suppose $d=-a$:** Then

$$\tau = \frac{a \pm \mathrm{i}}{c}, \qquad \text{with} \quad -bc = 1+a^2.$$

Again, since we care about the class of $A$ in $\mathrm{PSL}(2,\mathbb{Z})$, we may assume $c>0$ and $b<0$.

If $a=0$, we get $\tau = \mathrm{i} \in \mathcal{H}$ and

$$\mathbb{Z}_2 \cong \mathrm{PSL}(2,\mathbb{Z})_i = \left\langle S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

**Claim 5.14.** *If $a \neq 0$, then $\tau = \frac{\mathrm{i}+a}{c} \in \mathcal{H}$ lies in the* $\mathrm{PSL}(2,\mathbb{Z})$*-orbit of* i. *Therefore, by Remark 5.3, its isotropy group is conjugate to that of* i.

**Proof.** It is well known (e.g. see [**Sil94**, Ch. I]) that $\mathrm{SL}(2,\mathbb{Z})$ is generated by

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and every point in $\mathcal{H}$ lies in the orbit of a point in the **fundamental domain**

$$D = \left\{ \tau \in \mathcal{H} : -\tfrac{1}{2} \leq \mathrm{Re}(\tau) < \tfrac{1}{2},\ |\tau| \geq 1 \right\};$$

see Figure 1.

Since trace is invariant under conjugation, $\frac{\mathrm{i}+a}{c}$ is in the orbit of another point $\frac{\mathrm{i}+a'}{c'}$ with $-1/2 \leq \frac{a'}{c'} < 1/2$ and $1/c' > \sqrt{3}/2$. This forces $c'=1$ and $a'=0$, which gives us i. $\square$

**(2) Suppose $a + d = 1$:** Then

$$\tau = \frac{2a - 1 \pm \sqrt{3}\,\mathrm{i}}{2c}, \qquad \text{with} \quad a^2 - a + 1 = -bc.$$

If $a = 0$ and $c > 0$, we get

$$\tau = \mu := \frac{-1 + \sqrt{3}\,\mathrm{i}}{2} \in \mathcal{H}$$

and

$$\mathbb{Z}_3 \cong \mathrm{PSL}(2,\mathbb{Z})_\mu = \left\langle A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\rangle.$$

Note that $\mu$ is a cubic root of 1.

**Claim 5.15.** *For other values of $a$ and $c$,*

$$\tau = \frac{2a - 1 + \sqrt{3}\,\mathrm{i}}{2c} \in \mathcal{H}$$

*lies in the* $\mathrm{PSL}(2,\mathbb{Z})$*-orbit of $\mu$. Therefore, by Remark 5.3, its isotropy group is conjugate to that of $\mu$.*

**Proof.** The proof is similar to the previous claim. □

□

**Exercise 5.8.** With notation as in the previous exercise, note that the matrices $S$ and $A$ are not in $\Gamma_2$. Therefore, the stabilizer of every point is trivial, and the action is free. Since the action is also proper, the quotient $\mathcal{H}/\Gamma_2$ is a smooth manifold of real dimension two.

A fundamental domain for the action of $\Gamma_2$ consists of six copies of the standard fundamental domain for $\mathrm{PSL}(2,\mathbb{Z})$ (c.f. [**SG69**, p. 442]). By drawing this domain and identifying the edges appropriately, one can show that the quotient space is a sphere with three punctures. □

**Exercise 5.9.** By Remark 5.2, we only need to verify that the action is free. Since $p$ and $q$ are coprime, both $\mathrm{e}^{2\pi i/p}$ and $\mathrm{e}^{2\pi iq/p}$ have order $p$. Thus, since at least one of $z_1$ or $z_2$ is nonzero in $S^3$, the action is free.

Since $S^3$ is simply-connected, it follows from Remark 5.12 that $\pi_1(L(p,q)) \cong \mathbb{Z}_p$. □

**Exercise 5.10.** The action of $\mathbb{Z}$ on $S^1 \subset \mathbb{C}$ by

$$e^{\mathrm{i}\theta} \cdot n = e^{\mathrm{i}(\theta + n\theta_0)},$$

where $\theta_0/2\pi$ is irrational is free but not proper; see Remark 5.2. □

*Chapter 6*

# Tangent bundle; part I

For any open subset $V$ of a real or complex affine space $A$, and for every point $p \in V$, the tangent space $T_pV = T_pA$ consists of all possible directions in which one can move starting from $p$. In other words, $T_pV$ is simply a translated copy of the model vector space $A$, with the origin shifted to $p$ – a notion made precise by the linear structure of $A$. The union of these tangent spaces forms the tangent bundle of $V$, which is simply the product space

$$TV := \bigsqcup_{p \in V} T_pV \cong V \times A.$$

If $\gamma\colon (-\varepsilon, \varepsilon) \to V$ is a differentiable parametrized curve with $\gamma(0) = p$, then the derivative

$$\dot{\gamma}(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} \in T_pV \cong A$$

is the tangent vector to $\gamma$ at $p$.

Note that this expression makes sense because the linear structure on $A$ allows us to subtract points and rescale vectors. In this section, we generalize these ideas to define the tangent bundle of smooth and holomorphic manifolds – a key step in extending calculus to curved (i.e., non-flat) spaces.

**Remark 6.1.** If $V \subset \mathbb{H}_m$ is an open set, we define $TV = V \times \mathbb{R}^m$ as before. The tangent space at each point is still the full vector space $\mathbb{R}^m$. At boundary points, however, not every vector can be realized as the tangent vector to a smooth path. This subtlety is typical for all manifolds with boundary.

**Remark 6.2.** On a complex vector space $\mathbb{C}^n$, or an open subset thereof, a complex curve is the image of a holomorphic map

$$\gamma\colon \Delta \to \mathbb{C}^n, \qquad \gamma(0) = p,$$

where $\Delta \subset \mathbb{C}$ is an open disk centered at 0 of some radius $\varepsilon > 0$. The complex tangent vector to $\gamma$ is defined by the same formula:

$$\dot\gamma(0) = \lim_{z\to 0} \frac{\gamma(z) - \gamma(0)}{z} \in T_p\mathbb{C}^n \cong \mathbb{C}^n.$$

When considered as real manifolds, holomorphic curves are examples of Riemann surfaces.

There are at least two standard ways to define the tangent bundle of a smooth or holomorphic manifold:

- The first is a coordinate-free approach that interprets tangent vectors (and vector fields) as derivations – that is, as linear operators acting on smooth (or holomorphic) functions.
- The second is a more concrete construction that builds the tangent bundle by gluing together local data from charts, in the spirit of Section 4.

The second approach is direct and well-suited for calculations. It is also essential for proving that the tangent bundle naturally inherits a smooth or holomorphic structure, depending on the context.

The first approach, though less computationally friendly, is equally important, as it gives a global, intrinsic definition of the tangent bundle. It will become especially valuable later, when we introduce the Lie derivative.

We begin with the second (chart-based) construction and then move on to the operator-based approach in the next lecture, which will require a more in-depth discussion.

**Definition 6.3.** Suppose

$$M := \coprod_{\alpha\in\mathcal{I}} V_\alpha/\sim$$

is a smooth or holomorphic manifold, as in (4.1), obtained by gluing affine pieces $\{V_\alpha \subset A_\alpha\}_{\alpha\in\mathcal{I}}$ along overlap regions $\{V_{\alpha,\beta}\}_{\alpha,\beta\in\mathcal{I}}$ via the identifications

$$x \sim y \quad \Leftrightarrow \quad x \in V_{\alpha,\beta},\ y \in V_{\beta,\alpha},\ y = \varphi_{\alpha\mapsto\beta}(x).$$

Then the **tangent bundle** of $M$ is defined as the quotient space

$$TM := \coprod_{\alpha\in\mathcal{I}} (V_\alpha \times A_\alpha)/\sim, \tag{6.1}$$

obtained by gluing the pieces $\{TV_\alpha = V_\alpha \times A_\alpha\}_{\alpha\in\mathcal{I}}$ along their overlaps

$$\{TV_{\alpha,\beta} = V_{\alpha,\beta} \times A_\alpha\}_{\alpha,\beta\in\mathcal{I}}$$

using the identifications

$$(x,v) \sim (y,w) \quad \Leftrightarrow \quad \begin{cases} (x,v) \in V_{\alpha,\beta} \times A_\alpha, \\ (y,w) \in V_{\beta,\alpha} \times A_\beta, \\ y = \varphi_{\alpha\mapsto\beta}(x), \\ w = D_x\varphi_{\alpha\mapsto\beta}(v), \end{cases}$$

where $d_x\varphi_{\alpha\mapsto\beta} \in \mathrm{Isom}(A_\alpha, A_\beta)$ is the derivative of $\varphi_{\alpha\mapsto\beta}$ at $x$ in the usual calculus sense:

$$d_x\varphi_{\alpha\mapsto\beta}(v) = \lim_{t\to 0} \frac{\varphi_{\alpha\mapsto\beta}(x+tv) - \varphi_{\alpha\mapsto\beta}(x)}{t}.$$

When $A_\alpha = \mathbb{R}^m$, for all $\alpha \in \mathcal{I}$, the derivative $d_x\varphi_{\alpha\mapsto\beta} \in \mathrm{GL}(m,\mathbb{R})$ is simply the Jacobian matrix of partial derivatives. Similarly, if $A_\alpha = \mathbb{C}^m$ and $\varphi_{\alpha\mapsto\beta}$ is holomorphic, then $d_x\varphi_{\alpha\mapsto\beta} \in \mathrm{GL}(m,\mathbb{C})$ is the matrix of holomorphic partial derivatives.

It follows directly from the construction that the local projection maps $\pi_\alpha\colon TV_\alpha \to V_\alpha$ are compatible on overlaps and glue to a global surjective projection

$$\pi\colon TM \to M.$$

**Lemma 6.4.** *For any triple $\alpha, \beta, \gamma \in \mathcal{I}$, the following cocycle condition holds on $V_{\alpha,\beta\gamma} \times A_\alpha$:*

$$d\varphi_{\alpha\mapsto\gamma} = d\varphi_{\beta\mapsto\gamma} \circ d\varphi_{\alpha\mapsto\beta}.$$

*Moreover, the space $TM$ is automatically Hausdorff. Therefore, depending on the context, the quotient space $TM$ in* (6.1) *is a smooth or holomorphic manifold, and $\pi$ is a smooth or holomorphic map onto $M$. For each point $p \in M$, the fiber*

$$T_pM := \pi^{-1}(p) \tag{6.2}$$

*is called the* **tangent space** *to $M$ at $p$; it is naturally a vector space and is (non-canonically) isomorphic to $\mathbb{R}^m$ or $\mathbb{C}^m$, depending on the context..*

**Proof.** The cocycle condition on derivatives is simply the chain rule. For every $x \in V_\alpha$, the fiber $\pi_\alpha^{-1}(x) = \{x\} \times A_\alpha$ is a vector space identified with $A_\alpha$. If $x \in V_\alpha$ is equivalent to $y \in V_\beta$, then

$$d_x\varphi_{\alpha\mapsto\beta}\colon \{x\} \times A_\alpha \longrightarrow \{y\} \times A_\beta$$

is a linear isomorphism (because $\varphi_{\alpha\mapsto\beta}$ is a diffeomorphism). Hence, it preserves the linear structure but not the specific identification with $A_\alpha$. We conclude that the isomorphism classes $T_pM$ of the fibers in local pieces have well-defined vector space structures. However, the particular identification of each $T_pM$ with $\mathbb{R}^m$ or $\mathbb{C}^m$ depends on the choice of a basis.

For two distinct elements $u, u' \in TM$, either $\pi(u) \neq \pi(u')$, or they are different vectors in the same tangent space $T_pM$. In either case, since both $M$ and $A_\alpha$ are Hausdorff, they can clearly be separated by disjoint open sets. $\square$

**Remark 6.5.** There is a mild subtlety with Definition 6.3, as it a priori depends on the choice of an atlas on $M$. Recall from Lemma 4.1 that the quotient presentations (4.1) are in one-to-one correspondence with pairs $(M, \mathcal{A})$ consisting of a manifold and a countable atlas on it. However, it is straightforward to show that the manifold $TM$ associated to such a pair $(M, \mathcal{A})$ via (6.1) depends only on $M$.

Given two such atlases $\mathcal{A}$ and $\mathcal{A}'$, we can construct a common **refinement** by taking intersections of their domains. Therefore, it suffices to prove the claim when $\mathcal{A}'$ is a refinement of $\mathcal{A}$, and every chart in $\mathcal{A}'$ is either a sub-chart of some chart in $\mathcal{A}$, or disjoint from it. In this case, it is easy to construct an isomorphism from the quotient space associated to $\mathcal{A}'$ to the one associated to $\mathcal{A}$. We leave the details to the reader. Later in this chapter, we introduce a coordinate-free definition of the tangent bundle and show that it agrees with the construction above. This also confirms that $TM$ is intrinsically associated to the manifold $M$ itself, independent of the atlas used.

**Remark 6.6.** In the construction of $TM$ above, we started from a quilted space interpretation of $M$ as in (4.1) and similarly built $TM$ by gluing local pieces, each of which is the tangent space of an affine open subset. Given a manifold $M$ and an atlas $\mathcal{A} = \{\varphi \colon U_\alpha \longrightarrow V_\alpha\}$, the tangent bundle $TM$ is still described as a quilted space obtained by gluing the local pieces $TV_\alpha$ along the overlaps using the transition functions $\varphi_{\alpha\mapsto\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ and their derivatives. Therefore, in what follows, $TU_\alpha$ should be understood either as $TV_\alpha$ or via the coordinate-free description in terms of derivations discussed below.

**Definition 6.7.** A **vector field** on $M$ is a section of the projection map $\pi \colon TM \to M$ – that is, a map $\xi \colon M \to TM$ such that $\pi \circ \xi = \mathrm{id}_M$. In other words, $\xi$ assigns to each point $x \in M$ a vector $\xi(x) \in T_xM$.

Locally, any section $\xi_\alpha$ of $TV_\alpha = V_\alpha \times A_\alpha \to V_\alpha$ is given by the graph of a function

$$\xi_\alpha(x) = (x, X_\alpha(x)), \qquad X_\alpha \colon V_\alpha \to A_\alpha.$$

This section is continuous, smooth, or holomorphic depending on whether the function $X_\alpha$ has the corresponding regularity.

Globally, a section $\xi$ corresponds to a collection of functions $\{X_\alpha\colon V_\alpha \to A_\alpha\}_{\alpha\in\mathcal{I}}$ that are compatible on overlaps with respect to the transition functions, in the sense that

$$X_\beta(y) = d_x\varphi_{\alpha\mapsto\beta}(X_\alpha(x)) \qquad \text{for all } \alpha,\beta\in\mathcal{I} \text{ and } y = \varphi_{\alpha\mapsto\beta}(x). \tag{6.3}$$

Depending on the context, a section is said to be **continuous**, **smooth**, or **holomorphic** if it is locally of that type.

**Remark 6.8.** Every smooth manifold admits a plethora of smooth vector fields. One can start with an arbitrary collection of local vector fields on charts and patch them together to define a global vector field using a partition of unity. On the other hand, for closed (i.e., compact without boundary) holomorphic manifolds, the space of holomorphic vector fields is finite-dimensional and may even be trivial. This reflects, in part, the fact that holomorphic partitions of unity do not exist.

**Exercise 6.9.** Recall from Section 2 and the solution to Exercise 3.5 that $\mathbb{RP}^n$ (respectively, $\mathbb{CP}^n$) can be covered by $n+1$ charts $\varphi_j\colon U_j\to V_j\cong\mathbb{R}^n$ (respectively, $\mathbb{C}^n$), for $j=0,\dots,n$, with transition maps given by

$$\varphi_{i\mapsto j} = \varphi_j\circ\varphi_i^{-1}\big((x_k)_{k\neq i}\big) = (y_k)_{k\neq j}, \quad \text{where} \quad y_k = \begin{cases} x_k/x_j & \text{if } k\neq i,\\ 1/x_j & \text{if } k=i.\end{cases}$$

Does the vector field

$$X_0(x) := x_1\partial_{x_1} + \cdots + x_n\partial_{x_n}$$

defined on $V_0$ extend smoothly (respectively, holomorphically) to a vector field on all of $\mathbb{RP}^n$ (respectively, $\mathbb{CP}^n$)? If so, what is its expression on the other charts $V_j$? (Here, $\partial_{x_i}$ denotes the constant coordinate vector field corresponding to the $i$-th standard basis vector $e_i$ of $\mathbb{R}^n$ or $\mathbb{C}^n$.)

**Exercise 6.10.** Recall that the 2-sphere $S^2\subset\mathbb{R}^3$ can be covered by two charts $\varphi_\pm\colon U_\pm\to V_\pm\cong\mathbb{R}^2$, with transition map

$$\varphi_{+\mapsto-}\colon V_{+,-}=\mathbb{R}^2\setminus\{0\}\to V_{-,+}=\mathbb{R}^2\setminus\{0\}, \quad x=(x_1,x_2)\mapsto\frac{1}{|x|^2}(x_1,x_2).$$

Which of the following vector fields on $V_+$ extend smoothly to all of $S^2$?

$$x_1\partial_{x_1}+x_2\partial_{x_2}, \qquad x_1\partial_{x_1}-x_2\partial_{x_2}, \qquad x_2\partial_{x_1}-x_1\partial_{x_2}.$$

**Exercise 6.11.** Show that the map $\varphi\colon\mathbb{R}^3\to\mathbb{R}^3$ defined by

$$\varphi(x,y,z) = (2y,-x,-xy+z)$$

is a diffeomorphism. Let $X = x\partial_x + y\partial_y$ be a vector field on $\mathbb{R}^3$. If the pushforward $d\varphi(X)$ is expressed in coordinates as

$$d\varphi(X) = a\partial_x + b\partial_y + c\partial_z,$$

find the coefficient functions $a$, $b$, and $c$.

**Exercise 6.12.** Show that the complex vector space of holomorphic vector fields on $\mathbb{CP}^1$ is 3-dimensional and find a basis.

**Remark 6.13.** In a future exercise, you will learn to prove that the complex vector space of holomorphic vector fields on any complex 2-torus is 1-dimensional. For closed Riemann surfaces (i.e., closed holomorphic manifolds of complex dimension 1) of genus $g > 1$, one can show that there are no nontrivial holomorphic vector fields. We do not yet have the tools to prove this.

## Solutions to exercises

**Exercise 6.9.** We compute the expression for $X_0$ in terms of the coordinates on $V_j$, for each $j = 1, \ldots, n$, over the overlap region

$$V_{j,0} = \{(y_k)_{k \neq j} \colon y_0 \neq 0\},$$

and check whether this expression extends smoothly or holomorphically to the entire $V_j$. Since $U_0$ is dense in projective space, such an extension, if it exists, would be unique. To do this, we compute the pushforward $d\varphi_{0 \mapsto j}(X_0)$. We have

$$d\varphi_{0\mapsto j}(X_0) = \sum_{i=1}^{n} x_i \; d\varphi_{0\mapsto j}(\partial_{x_i}) = \sum_{i=1}^{n} x_i \sum_{k \neq j} \frac{\partial y_k}{\partial x_i} \; \partial_{y_k}.$$

The partial derivatives are given by

$$\frac{\partial y_k}{\partial x_i} = \begin{cases} 1/x_j & \text{if } i = k \neq 0, \\ -x_k/x_j^2 & \text{if } k \neq 0, i = j, \\ 0 & \text{if } k \neq 0, i \neq k, j, \\ -1/x_j^2 & \text{if } k = 0, i = j, \\ 0 & \text{if } k = 0, i \neq j. \end{cases}$$

Therefore,

$$\begin{aligned} d\varphi_{0\mapsto j}(X_0) &= \sum_{i \neq j,0} x_i \frac{\partial y_i}{\partial x_i} \; \partial_{y_i} + \sum_{i \neq j,0} x_j \frac{\partial y_i}{\partial x_j} \; \partial_{y_i} + \sum_{i=1}^{n} x_i \frac{\partial y_0}{\partial x_i} \; \partial_{y_0} \\ &= \sum_{i \neq j,0} \frac{x_i}{x_j} \; \partial_{y_i} - \sum_{i \neq j,0} \frac{x_i}{x_j} \; \partial_{y_i} - \frac{1}{x_j} \; \partial_{y_0} = -\frac{1}{x_j} \; \partial_{y_0} = -y_0 \; \partial_{y_0} \end{aligned}$$

It is clear that this final expression extends smoothly (or holomorphically) to the entirety of $V_j$, depending on context. Thus, the local vector field $X_0$ on $V_0$, together with the local vector fields $X_j = -y_0 \; \partial_{y_0}$ on $V_j$ for all $j = 1, \ldots, n$, are compatible on overlaps and define a global vector field $\xi$ on projective space. □

**Exercise 6.10.** Let $y = (y_1, y_2)$ denote the coordinates on $V_-$. *It is always good practice to distinguish between the coordinates on the domain and those on the target*, since in computing the pushforward we eventually need to express coefficients with respect to the target coordinates.

Since

$$(y_1, y_2) = \frac{1}{x_1^2 + x_2^2}(x_1, x_2),$$

we compute

$$d\varphi_{+\mapsto -} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 \end{bmatrix} = \frac{1}{|x|^4}\begin{bmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{bmatrix}.$$

We find that

$$\begin{aligned}
d\varphi_{+\mapsto -}(x_1\partial_{x_1} + x_2\partial_{x_2}) =& \frac{1}{|x|^4}\begin{bmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
=& \frac{1}{|x|^2}\begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} -y_1 \\ -y_2 \end{bmatrix}; \\
d\varphi_{+\mapsto -}(x_1\partial_{x_1} - x_2\partial_{x_2}) =& \frac{1}{|x|^4}\begin{bmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} \\
=& \frac{1}{|x|^4}\begin{bmatrix} -x_1^3 + 3x_1x_2^2 \\ x_2^3 - 3x_2x_1^2 \end{bmatrix}; \\
d\varphi_{+\mapsto -}(x_2\partial_{x_1} - x_1\partial_{x_2}) =& \frac{1}{|x|^4}\begin{bmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{bmatrix}\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \\
=& \frac{1}{|x|^2}\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix}.
\end{aligned}$$

The first equation shows that the local vector field $x_1\partial_{x_1} + x_2\partial_{x_2}$ on $V_+$ matches the local vector field $-y_1\partial_{y_1} - y_2\partial_{y_2}$ on $V_-$. Together they define a global vector field on $S^2$ that vanishes at two antipodal points – one a source (where vectors point outward) and the other a sink (where vectors point inward).

The third equation shows that the local vector field $x_2\partial_{x_1} - x_1\partial_{x_2}$ on $V_+$ matches the local vector field $y_2\partial_{y_1} - y_1\partial_{y_2}$ on $V_-$. These together define a global vector field on $S^2$ that also vanishes at two antipodal points. This vector field corresponds to rotation around the axis passing through those points.

In the second equation, we also note:

$$\frac{1}{|x|^4}\begin{bmatrix} -x_1^3 + 3x_1x_2^2 \\ x_2^3 - 3x_2x_1^2 \end{bmatrix} = |y|^{-2}\begin{bmatrix} -y_1^3 + 3y_1y_2^2 \\ y_2^3 - 3y_2y_1^2 \end{bmatrix},$$

and ask whether the rational functions

$$\frac{y_1(3y_2^2 - y_1^2)}{y_1^2 + y_2^2} \qquad \text{and} \qquad \frac{y_2(y_2^2 - 3y_1^2)}{y_1^2 + y_2^2}$$

admit continuous or smooth extensions to the origin $y = (0,0)$. In polar coordinates, we have

$$\frac{y_1(3y_2^2 - y_1^2)}{y_1^2 + y_2^2} = r\cos(\theta)(4\sin^2(\theta) - 1),$$
$$\frac{y_2(y_2^2 - 3y_1^2)}{y_1^2 + y_2^2} = -r\sin(\theta)(4\cos^2(\theta) - 1).$$

Thus, the vector field

$$|y|^{-2}\begin{bmatrix} -y_1^3 + 3y_1y_2^2 \\ y_2^3 - 3y_2y_1^2 \end{bmatrix}$$

extends continuously to the origin $r = 0$ by the zero vector. However, for instance,

$$\frac{\partial}{\partial y_2}\left(\frac{3y_2^2y_1 - y_1^3}{y_1^2 + y_2^2}\right) = \frac{6y_1y_2|y|^2 - 6y_2^3y_1 + 2y_2y_1^3}{|y|^4} = 8\sin(\theta)\cos(\theta)^3,$$

which does not extend continuously to the origin. We conclude that the local vector field $x_1\partial_{x_1} + x_2\partial_{x_2}$ on $V_+$ defines a continuous vector field on all of $S^2$, but the extension fails to be $C^1$ at one point.

□

**Exercise 6.11.** For the sake of clarity, let us denote the target coordinates by $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$. Then,

$$(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = (2y, -x, -xy + z),$$

so that

$$(x, y, z) = (-\mathfrak{y}, \frac{\mathfrak{x}}{2}, \mathfrak{z} - \frac{\mathfrak{x}\mathfrak{y}}{2}).$$

Therefore, $\varphi$ is a diffeomorphism. We have

$$\begin{aligned} d\varphi(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) &= x\left\{\frac{\partial\mathfrak{x}}{\partial x}\partial_\mathfrak{x} + \frac{\partial\mathfrak{y}}{\partial x}\partial_\mathfrak{y} + \frac{\partial\mathfrak{z}}{\partial x}\partial_\mathfrak{z}\right\} + y\left\{\frac{\partial\mathfrak{x}}{\partial y}\partial_\mathfrak{x} + \frac{\partial\mathfrak{y}}{\partial y}\partial_\mathfrak{y} + \frac{\partial\mathfrak{z}}{\partial y}\partial_\mathfrak{z}\right\} \\ &= \mathfrak{x}\partial_\mathfrak{x} + \mathfrak{y}\partial_\mathfrak{y} + \mathfrak{x}\mathfrak{y}\partial_\mathfrak{z}. \end{aligned}$$

Switching the notation back from $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$ to $(x, y, z)$, we get

$$a(x, y, z) = x, \quad b(x, y, z) = y, \quad c(x, y, z) = xy.$$

□

**Exercise 6.12.** Recall from the solution to Exercise 2.11 that $\mathbb{CP}^1$ can be covered by two copies of $\mathbb{C}$, $V_0 = \mathbb{C}$ and $V_1 = \mathbb{C}$, with the following gluing data:

- $V_{0,1}, V_{1,0} = \mathbb{C}^*$;
- the transition map $\varphi_{0\mapsto 1}\colon \mathbb{C}^* \to \mathbb{C}^*$ is given by $z \mapsto w = z^{-1}$, where $z$ is the coordinate on $V_0$ and $w$ is the coordinate on $V_1$.

Therefore, a holomorphic vector field on $\mathbb{CP}^1$ is given by a collection of two local holomorphic vector fields $f(z)\partial_z$ and $g(w)\partial_w$ on $V_0$ and $V_1$, respectively, such that

$$g(w)\partial_w = d\varphi_{0\mapsto 1}(f(z)\partial_z).$$

We have

$$d\varphi_{0\mapsto 1}(f(z)\partial_z) = f(z)\ d\varphi_{0\mapsto 1}(\partial_z) = -f(z)z^{-2}\partial_w.$$

Therefore, the compatibility condition becomes

$$-z^2 g(z^{-1}) = f(z).$$

Since $f$ and $g$ are holomorphic functions on the entire plane, they have everywhere convergent Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(w) = \sum_{n=0}^{\infty} b_n w^n.$$

Using these expressions, the compatibility equation reads

$$-\sum_{n=0}^{\infty} b_n z^{2-n} = \sum_{n=0}^{\infty} a_n z^n.$$

We conclude that both $f(z)$ and $g(w)$ are quadratic polynomials and

$$a_0 = -b_2, \quad a_1 = -b_1, \quad a_2 = -b_0.$$

Hence, the complex vector space of holomorphic vector fields on $\mathbb{CP}^1$ is 3-dimensional and has basis $\partial_z, z\partial_z, z^2\partial_z$.

Chapter 7

# Tangent bundle; part II

In this lecture, we explore the identification of tangent spaces and vector fields with corresponding spaces of derivations.

For every open subset $V$ of an affine space $A$ (or of $\mathbb{H}_m$), let $C^\infty(V,\mathbb{R})$ denote the space of smooth real-valued functions on $V$. Every vector field $X$ on $V$ defines an $\mathbb{R}$-linear operator

$$D_X \colon C^\infty(V,\mathbb{R}) \to C^\infty(V,\mathbb{R})$$

by the formula

$$D_X(f) := X \cdot f := df(X) = \lim_{t\to 0} \frac{f(x+tX) - f(x)}{t}.$$

In local coordinates $(x_1,\ldots,x_m)$, if $X = \sum_{i=1}^m a_i(x)\,\partial_{x_i}$, then

$$X \cdot f = df\left(\sum_{i=1}^m a_i(x)\,\partial_{x_i}\right) = \sum_{i=1}^m a_i(x)\frac{\partial f}{\partial x_i}.$$

While the limit above does not make sense on abstract manifolds, the notion of a "derivation" provides an intrinsic generalization. We will later prove that there is a one-to-one correspondence between derivations and vector fields in the sense of Definition 6.7.

**Definition 7.1.** Let $M$ be a smooth manifold. Denote by $C^\infty(M,\mathbb{R})$ the space of smooth functions $M \to \mathbb{R}$. A **derivation** on $M$ is an $\mathbb{R}$-linear map

$$D\colon C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$$

satisfying the Leibniz rule:

$$D(fg) = fD(g) + gD(f) \qquad \forall\, f,g \in C^\infty(M,\mathbb{R}).$$

Similarly, if $M$ is a holomorphic manifold, let $C^{\mathrm{hol}}(M,\mathbb{C})$ denote the space of holomorphic functions $M\to\mathbb{C}$. A **holomorphic derivation** on $M$ is a $\mathbb{C}$-linear map

$$D\colon C^{\mathrm{hol}}(M,\mathbb{C}) \longrightarrow C^{\mathrm{hol}}(M,\mathbb{C})$$

also satisfying the Leibniz rule. We denote the space of smooth and holomorphic derivations by $\mathrm{Der}_{C^\infty}(M)$ and $\mathrm{Der}_{\mathrm{hol}}(M)$, respectively.

**Remark 7.2.** For closed holomorphic manifolds, the space $C^{\mathrm{hol}}(M,\mathbb{C})$ consists only of constant functions. As a result, $\mathrm{Der}_{\mathrm{hol}}(M)$ is trivial and not an interesting object to study.

**Definition 7.3.** Let $M$ be a smooth manifold. Two smooth functions $f\colon U\to\mathbb{R}$ and $g\colon U'\to\mathbb{R}$, defined on neighborhoods of $p\in M$, are said to have the same **germ at** $p$ if they agree on some smaller neighborhood $U''\subset U\cap U'$.

Similarly, if $M$ is a holomorphic manifold, two holomorphic functions $f\colon U\to\mathbb{C}$ and $g\colon U'\to\mathbb{C}$ have the same germ at $p$ if they coincide on some neighborhood $U''\subset U\cap U'$.

For any $p\in M$, having the same germ is an equivalence relation on the set of smooth (or holomorphic) functions defined near $p$. Each equivalence class is called the **germ** of a smooth (or holomorphic) function at $p$. The set of all such germs is denoted by $C_p^\infty(M,\mathbb{R})$ or $C_p^{\mathrm{hol}}(M,\mathbb{C})$, respectively. These are rings under point-wise addition and multiplication. The operations are well-defined because representative functions can always be restricted to a common domain without affecting the germ. The property of vanishing at a point is well-defined for germs of functions at that point. In both the smooth and holomorphic cases, we denote by $I_p$ the ideal in $C_p^\infty(M,\mathbb{R})$ or $C_p^{\mathrm{hol}}(M,\mathbb{C})$ consisting of germs of functions that vanish at $p$.

**Remark 7.4.** In the holomorphic case, if $f$ and $g$ have the same germ at $p$, and $U$ and $U'$ are connected, then there exists a holomorphic function on the union $U\cup U'$ that restricts to $f$ on $U$ and to $g$ on $U'$. In other words, every holomorphic function on a connected domain is uniquely determined by its germ at any point. Thus, the situation is much more rigid than in the smooth case.

**Definition 7.5.** Let $M$ be a smooth manifold. A **derivation at** $p\in M$ is an $\mathbb{R}$-linear map

$$D\colon C_p^\infty(M,\mathbb{R})\to\mathbb{R}$$

satisfying the Leibniz rule:

$$D(fg)=f(p)D(g)+g(p)D(f) \qquad \forall\, f,g\in C_p^\infty(M,\mathbb{R}).$$

We denote the space of such derivations by $\mathrm{Der}_{C^\infty}(M,p)$.

Similarly, if $M$ is a holomorphic manifold, a **holomorphic derivation at** $p$ is a $\mathbb{C}$-linear map

$$D\colon C_p^{\mathrm{hol}}(M,\mathbb{C}) \to \mathbb{C}$$

satisfying the Leibniz rule. The space of such derivations is denoted by $\mathrm{Der}_{\mathrm{hol}}(M,p)$.

**Theorem 7.6.** *(i) For every point $p \in M$, there are canonical isomorphisms of vector spaces*

$$T_pM \cong \mathrm{Der}_\star(M,p) \cong (I_p/I_p^2)^*,$$

*where $\star$ denotes either $C^\infty$ or* hol*, depending on whether $M$ is a smooth or holomorphic manifold.*

*(ii) If $M$ is smooth, there is $C^\infty(M,\mathbb{R})$-module isomorphism between the space of smooth vector fields on $M$ and $\mathrm{Der}_{C^\infty}(M)$.*

**Remark 7.7.** A statement analogous to Theorem 7.6.(ii) holds for holomorphic manifolds; however, the proof given below does not extend to this setting, as holomorphic manifolds do not admit compactly supported functions. In algebraic geometry, too, the analogue of the quotient $(I_p/I_p^2)^*$ plays the role of the tangent space at a point; c.f [**Har77**, Ch. II.8]

**Proof.** Part (i).

**Claim A.** For every derivation $D$ on $M$ and every constant function $c$, we have $D(c) \equiv 0$. The same holds for derivations at a point $p$ and the germ of a constant function $c$ at $p$.

**Proof of Claim A.** By the Leibniz rule,

$$D(1) = D(1 \cdot 1) = D(1) + D(1).$$

Therefore, $D(1) = 0$. By $\mathbb{R}$-linearity (or $\mathbb{C}$-linearity), we have $D(c) = cD(1) = 0$. □

**Claim B.** Every $D \in \mathrm{Der}_\star(M,p)$ vanishes on $I_p^2$.

**Proof of Claim B.** The ideal $I_p^2$ consists of real or complex linear combinations of products of elements in $I_p$. So it suffices to show that $D(fg) = 0$ for all $f, g \in I_p$. By the Leibniz rule,

$$D(fg) = f(p)D(g) + g(p)D(f) = 0.$$

□

**Conclusion I.** By Claim B, every derivation $D$ at $p$ descends to a well-defined linear map

$$D\colon I_p/I_p^2 \longrightarrow \mathbb{R} \quad \text{or} \quad \mathbb{C},$$

depending on the context. Furthermore, by Claim A, $D$ is uniquely determined by its action on $I_p$. That is, we obtain a canonical injective linear map

$$\mathrm{Der}_\star(M,p) \longrightarrow (I_p/I_p^2)^*.$$

Conversely, any linear map on $I_p/I_p^2$ lifts to a derivation at $p$. Hence, we obtain a canonical isomorphism

$$\mathrm{Der}_\star(M,p) \cong (I_p/I_p^2)^*.$$

**Claim C.** Every $v \in T_pM$ in the sense of (6.2) naturally defines a derivation $D_v$ at $p$, giving a linear map

$$T_pM \longrightarrow \mathrm{Der}_\star(M,p).$$

**Proof of Claim C.** Let $\varphi\colon U \longrightarrow V$ be a chart around $p$, sending $p$ to $0 \in V$. Then, by definition, $v$ corresponds to a vector in $T_0V$. For any smooth or holomorphic function $f$ defined on a neighborhood of $p$, define

$$D_v(f) = \lim_{t\to 0} \frac{(f\circ\varphi^{-1})(tv) - (f\circ\varphi^{-1})(0)}{t}.$$

This limit exists by smoothness or holomorphicity of $f$, and $D_v$ satisfies the Leibniz rule. □

**Remark 7.8.** The assignment $v \mapsto D_v$ depends on the choice of chart, but different charts yield the same operator $D_v$ by the chain rule and the definition of $T_pM$.

**Claim D.** Suppose $U$ is an open neighborhood of $p$ and $\varphi\colon U \to B_\varepsilon(0) \subset \mathbb{R}^m$ (or $\mathbb{C}^m$) is a chart (in the smooth or holomorphic atlas of $M$) with $\varphi(p) = 0$. Then for every function $f$ defined near $p$,

$$f\circ\varphi^{-1}(x_1,\dots,x_m) = f(p) + \sum_{i=1}^m x_i g_i(x), \tag{7.1}$$

where the functions $g_i$ are smooth or holomorphic (depending on context), and

$$g_i(0) = \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}(0).$$

**Proof of Claim D.** In the holomorphic case, the result follows directly from the Taylor expansion. We give the proof in the smooth case.

For $x \in B_\varepsilon(0)$, by the Fundamental Theorem of Calculus (first equality) and the chain rule (second), we have

$$\begin{aligned} f \circ \varphi^{-1}(x) - f \circ \varphi^{-1}(0) &= \int_0^1 \frac{d}{dt}\big(f \circ \varphi^{-1}(tx)\big)\, dt \\ &= \int_0^1 \sum_{i=1}^m x_i \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(tx)\, dt \\ &= \sum_{i=1}^m x_i \int_0^1 \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(tx)\, dt. \end{aligned}$$

Define

$$g_i(x) = \int_0^1 \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(tx)\, dt \qquad \text{for } i = 1, \ldots, m.$$

Each $g_i$ is smooth, and the formula (7.1) holds. Moreover,

$$g_i(0) = \int_0^1 \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(0)\, dt = \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(0).$$

**Conclusion II.** Applying any derivation $D$ at $p$ to (7.1), we get

$$D(f) = \sum_{i=1}^m \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(0) \cdot D(x_i).$$

Here, we treat the $i$-th coordinate $x_i$ as function from $U$ to $\mathbb{R}$ or $\mathbb{C}$. Let $a_i = D(x_i) \in \mathbb{R}$ or $\mathbb{C}$, and define

$$v = \sum_{i=1}^m a_i \partial_{x_i} \in T_0 V.$$

Then $D_v(f) = D(f)$ for every $f$ in $C_p^\infty(M, \mathbb{R})$ or $C_p^{\mathrm{hol}}(M, \mathbb{C})$, depending on context. This gives an inverse

$$\text{(7.2)} \qquad \mathrm{Der}_\star(M, p) \longrightarrow T_p M, \qquad D \mapsto \sum_{i=1}^m D(x_i)\partial_{x_i}$$

to the linear map in Claim C. Therefore, we obtain a canonical isomorphism

$$T_p M \cong \mathrm{Der}_\star(M, p).$$

These steps complete the proof of part (i) of Theorem 7.6.

For part (ii), in one direction, we show that every $D \in \mathrm{Der}_{C^\infty}(M)$ induces a derivation $D_p$ at each point $p \in M$. By part (i), $D_p$ is equal to $D_{v(p)}$ for some $v(p) \in T_p M$. Thus, the collection $\{v(p)\}_{p \in M}$ defines a vector field $\xi$ on $M$. Moreover, it follows from (7.2) that the coefficients of $\xi$ in any chart are smooth; hence, $\xi$ is a smooth vector field.

Suppose $f$ is a smooth function defined on an open set $U$ containing $p$. Choose an open subset $U' \subset U$ such that $p \in U'$ and $\mathrm{cl}_M U' \subset U$ is compact. Then there exists a compactly supported smooth function $\varrho\colon U \to [0,1]$ such that $\varrho|_{U'} \equiv 1$. The function $\varrho f$ can be extended by zero to all of $M$ and agrees with $f$ in a neighborhood of $p$, so it defines the same germ at $p$ as $f$. We define the induced derivation at $p$ by

$$D_p(f) := \big(D(\varrho f)\big)(p),$$

which satisfies the Leibniz rule at $p$ and is well-defined.

In the other direction, given a vector field $\xi$ as defined in Definition 6.7, it corresponds to a collection of local vector fields $\{X_\alpha\colon V_\alpha \to A_\alpha\}_{\alpha\in\mathcal{I}}$ that are compatible on overlaps with respect to the transition functions, in the sense that

$$X_\beta(y) = D_x\varphi_{\alpha\mapsto\beta}(X_\alpha(x)) \qquad \text{for all } \alpha,\beta \in \mathcal{I} \text{ and } y = \varphi_{\alpha\mapsto\beta}(x).$$

Similarly, every smooth function $f\colon M \to \mathbb{R}$ is represented by a compatible collection of functions $\{f_\alpha\colon V_\alpha \to \mathbb{R}\}_{\alpha\in\mathcal{I}}$, in the sense that

$$f_\alpha = f_\beta \circ \varphi_{\alpha\mapsto\beta} \qquad \text{for all } \alpha,\beta \in \mathcal{I}.$$

Define

$$\widetilde{f}_\alpha = X_\alpha \cdot f_\alpha \qquad \text{for all } \alpha \in \mathcal{I}.$$

It follows from the chain rule that the collection $\{\widetilde{f}_\alpha\colon V_\alpha \to \mathbb{R}\}_{\alpha\in\mathcal{I}}$ is also compatible, and thus defines a global smooth function $\widetilde{f}\colon M \to \mathbb{R}$. We define the derivation associated to $\xi$ by

$$D_\xi(f) := \widetilde{f},$$

as desired.

It is easy to verify that $D_{g\xi} = gD_\xi$. Therefore, the map

$$\xi \longrightarrow D_\xi$$

define a $C^\infty(M,\mathbb{R})$-module isomorphism between the space of smooth vector fields on $M$ and $\mathrm{Der}_{C^\infty}(M)$.

□

**Exercise 7.9.** Every open set $U \subset M$ of a smooth manifold $M$ is itself a smooth manifold. Therefore, Definition 7.1 applies to $U$. Show that if $U \subset U'$ are open subsets of $M$, then there is a canonical restriction map

$$\mathrm{Der}_{C^\infty}(U') \longrightarrow \mathrm{Der}_{C^\infty}(U).$$

For every pair of open subsets $U_1$ and $U_2$, suppose $D_1 \in \mathrm{Der}_{C^\infty}(U_1)$ and $D_2 \in \mathrm{Der}_{C^\infty}(U_2)$ have the same restriction to $U_1 \cap U_2$. Show that there exists $D \in \mathrm{Der}_{C^\infty}(U_1 \cup U_2)$ such that $D|_{U_i} = D_i$ for $i = 1, 2$.

**Definition 7.10.** Given a smooth manifold $M$ and two derivations $D_1, D_2 \in \mathrm{Der}_{C^\infty}(M)$, their **commutator** is the operator

$$\begin{aligned}&[D_1, D_2]\colon C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R}),\\ &[D_1, D_2](f) = D_1(D_2(f)) - D_2(D_1(f)) \quad \forall\, f \in C^\infty(M,\mathbb{R}).\end{aligned}$$

**Lemma 7.11.** *For all $D_1, D_2 \in \mathrm{Der}_{C^\infty}(M)$, the commutator $[D_1, D_2]$ is also a derivation on $M$.*

**Proof.** The commutator of two $\mathbb{R}$-linear maps is $\mathbb{R}$-linear. We must verify the Leibniz rule. We compute:

$$\begin{aligned}[D_1, D_2](fg) &= D_1(D_2(fg)) - D_2(D_1(fg))\\ &= D_1(fD_2(g) + gD_2(f)) - D_2(fD_1(g) + gD_1(f))\\ &= fD_1(D_2(g)) + D_1(f)D_2(g) + D_1(g)D_2(f) + gD_1(D_2(f))\\ &\quad - D_2(f)D_1(g) - fD_2(D_1(g)) - D_2(g)D_1(f) - gD_2(D_1(f))\\ &= f\big(D_1(D_2(g)) - D_2(D_1(g))\big) + g\big(D_1(D_2(f)) - D_2(D_1(f))\big)\\ &= f[D_1, D_2](g) + g[D_1, D_2](f).\end{aligned}$$

This confirms that $[D_1, D_2]$ satisfies the Leibniz rule. ☐

**Corollary 7.12.** *For every pair of smooth vector fields $\zeta, \xi$ on $M$, there exists a vector field $[\zeta, \xi]$ on $M$, called the* **Lie bracket of $\zeta$ and $\xi$**, *such that*

$$[\zeta, \xi] \cdot f = \zeta \cdot (\xi \cdot f) - \xi \cdot (\zeta \cdot f) \quad \forall\, f \in C^\infty(M,\mathbb{R}).$$

**Proof.** This follows from the lemma above and Theorem 7.6.(ii). ☐

**Exercise 7.13.** In local coordinates $(x_1, \dots, x_m)$, suppose

$$X_1 = \sum_{i=1}^m a_i(x)\, \partial_{x_i} \quad \text{and} \quad X_2 = \sum_{i=1}^m b_i(x)\, \partial_{x_i}.$$

Compute the coefficients of the expansion of the Lie bracket $[X_1, X_2]$ in terms of the functions $a_i(x)$ and $b_i(x)$.

**Exercise 7.14.** Let

$$X_1 = \sum_{i=1}^n x_i \partial_{y_i} \quad \text{and} \quad X_2 = \sum_{i=1}^n y_i \partial_{x_i}$$

be vector fields on $\mathbb{R}^{2n}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$. Compute $[X_1, X_2]$.

## Solutions to exercises

**Exercise 7.9.** Starting with any $D \in \mathrm{Der}_{C^\infty}(U')$, by Theorem 7.6.(ii), there is a vector field $\xi$ on $U'$ such that $D(f) = \xi \cdot f$ for all $f \in C^\infty(U', \mathbb{R})$. Define $D|_U \in \mathrm{Der}_{C^\infty}(U)$ to be the derivation corresponding to the restricted vector field $\xi|_U$.

**Remark 7.15.** A direct description of the restriction $D|_U$ is possible but takes longer to describe, because not every $f \in C^\infty(U, \mathbb{R})$ extends smoothly to $U'$. In order to directly define $D|_U$ by its action on $f$, we can describe the output function

$$D|_U(f) = g$$

as follows. For every point $p \in U$, choose a bump function $\varrho$ supported in $U$ that is equal to 1 near $p$ and such that $\varrho f$ extends smoothly to $U'$. Then define $g(p)$ to be the value of $D(\varrho f)$ at $p$. With this definition, one must check that $g$ is independent of the choice of $\varrho$ and that it satisfies the Leibniz rule. The approach used above via vector fields is clearly simpler.

If $\xi_1$ and $\xi_2$ are the vector fields corresponding to $D_1$ and $D_2$, then $\xi_1|_{U_1 \cap U_2} = \xi_2|_{U_1 \cap U_2}$. Therefore, they patch together to define a vector field $\xi$ on $U_1 \cup U_2$, which determines a derivation $D$ on $U_1 \cup U_2$ such that $D|_{U_i} = D_i$. $\square$

**Exercise 7.13.** Since partial derivatives commute, we have

$$[\partial_{x_i}, \partial_{x_j}] = 0 \qquad \forall\ 1 \leq i, j \leq m.$$

Also, it is easy to show that

$$[fX_1, X_2] = f[X_1, X_2] - (X_2 \cdot f)\, X_1, \qquad [X_1, fX_2] = f[X_1, X_2] + (X_1 \cdot f)\, X_2.$$

Therefore,

$$\begin{aligned}
\left[\sum_{i=1}^m a_i(x)\, \partial_{x_i}, \sum_{j=1}^m b_j(x)\, \partial_{x_j}\right] &= \sum_{i=1}^m \sum_{j=1}^m \Big[a_i(x)\, \partial_{x_i}, b_j(x)\, \partial_{x_j}\Big] \\
&= \sum_{i=1}^m \sum_{j=1}^m \left(a_i\, [\partial_{x_i}, b_j\, \partial_{x_j}] - b_j\, \frac{\partial a_i}{\partial x_j}\, \partial_{x_i}\right) \\
&= \sum_{i=1}^m \sum_{j=1}^m \left(a_i\, \frac{\partial b_j}{\partial x_i}\, \partial_{x_j} - b_j\, \frac{\partial a_i}{\partial x_j}\, \partial_{x_i}\right) \\
&= \sum_{i=1}^m \left(\sum_{j=1}^m a_j\, \frac{\partial b_i}{\partial x_j} - b_j\, \frac{\partial a_i}{\partial x_j}\right) \partial_{x_i}.
\end{aligned}$$

$\square$

**Exercise 7.14.** Similarly to the previous exercise, we have

$$\begin{aligned}
\left[\sum_{i=1}^{n} x_i\,\partial_{y_i}, \sum_{j=1}^{n} y_j\,\partial_{x_j}\right] &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left[x_i\,\partial_{y_i}, y_j\,\partial_{x_j}\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i\,[\partial_{y_i}, y_j\,\partial_{x_j}] - y_j\,\frac{\partial x_i}{\partial x_j}\,\partial_{y_i}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i\,\frac{\partial y_j}{\partial y_i}\,\partial_{x_j} - y_j\,\frac{\partial x_i}{\partial x_j}\,\partial_{y_i}\right) \\
&= \sum_{i=1}^{n}\left(x_i\,\partial_{x_i} - y_i\,\partial_{y_i}\right).
\end{aligned}$$

$\square$

*Chapter 8*

# Regular level sets

In this lecture, we define and study the smooth (respectively, holomorphic) derivative

$$df\colon TM \longrightarrow TM'$$

of a smooth (respectively, holomorphic) map $f\colon M \longrightarrow M'$, which opens up many possibilities for doing interesting things with manifolds. The derivative

$$df\colon TM \longrightarrow TM'$$

is a lift of $f$ that, depending on the context, maps the fiber $T_pM$ to $T_{f(p)}M'$ by a real or complex linear map, for every point $p \in M$.

Consider smooth or holomorphic atlases

$$\mathcal{A} = \big\{\varphi_\alpha\colon U_\alpha \longrightarrow V_\alpha \subset A_\alpha\big\}_{\alpha\in\mathcal{I}} \text{ and } \mathcal{A}' = \big\{\varphi_{\alpha'}\colon U'_{\alpha'} \longrightarrow V'_{\alpha'} \subset A'_{\alpha'}\big\}_{\alpha'\in\mathcal{I}'}$$

on $M$ and $M'$, respectively. From the perspective of Section 6, the tangent bundle $TM$ is constructed by gluing together the local models $TV_\alpha \cong V_\alpha \times A_\alpha$ via transition maps

$$V_{\alpha,\beta} \times A_\alpha \longrightarrow V_{\beta,\alpha} \times A_\beta, \qquad (x,v) \mapsto \big(\varphi_{\alpha\mapsto\beta}(x), D_x\varphi_{\alpha\mapsto\beta}(v)\big).$$

The construction of $TM'$ is similar. Also, from this point of view, a map $f\colon M \longrightarrow M'$ corresponds to a collection of maps

$$f_{\alpha\mapsto\alpha'}\colon V_{\alpha;\alpha'} \subset V_\alpha \longrightarrow V_{\alpha'} \qquad \forall\, \alpha \in \mathcal{I},\ \alpha' \in \mathcal{I}',$$

satisfying the compatibility condition

$$\varphi_{\alpha'\mapsto\beta'} \circ f_{\alpha\mapsto\alpha'} = f_{\beta\mapsto\beta'} \circ \varphi_{\alpha\mapsto\beta} \qquad \forall\, \alpha,\beta \in \mathcal{I},\ \alpha',\beta' \in \mathcal{I}'; \tag{8.1}$$

see Section 4. The derivative $df$ is then given by a collection of maps

$$df_{\alpha\mapsto\alpha'}\colon TV_{\alpha;\alpha'} = V_{\alpha;\alpha'} \times A_\alpha \longrightarrow V_{\alpha'} \times A'_{\alpha'} \qquad \forall\, \alpha \in \mathcal{I},\ \alpha' \in \mathcal{I}', \tag{8.2}$$

where each $df_{\alpha\mapsto\alpha'}$ is the usual derivative in the calculus sense:

$$df_{\alpha\mapsto\alpha'}(x,v) = \big(y = f_{\alpha\mapsto\alpha'}(x),\ w\big),$$
$$w = \lim_{t\to 0}\frac{f_{\alpha\mapsto\alpha'}(x+tv) - f_{\alpha\mapsto\alpha'}(x)}{t} \in A'_{\alpha'}.$$

When $A_\alpha = \mathbb{R}^m$ and $A'_{\alpha'} = \mathbb{R}^{m'}$, the derivative $d_x f_{\alpha\mapsto\alpha'}$ is represented by an $m' \times m$ matrix of partial derivatives. If instead $A_\alpha = \mathbb{C}^m$, $A'_{\alpha'} = \mathbb{C}^{m'}$, and $f_{\alpha\mapsto\alpha'}$ is holomorphic, then $d_x f_{\alpha\mapsto\alpha'}$ is a matrix in $\mathrm{M}_{m'\times m}(\mathbb{C})$ consisting of holomorphic partial derivatives.

Applying the chain rule to equation (8.1) gives

$$d\varphi_{\alpha'\mapsto\beta'} \circ df_{\alpha\mapsto\alpha'} = df_{\beta\mapsto\beta'} \circ d\varphi_{\alpha\mapsto\beta} \qquad \forall\ \alpha,\beta \in \mathcal{I},\ \alpha',\beta' \in \mathcal{I}'.$$

This shows that the collection of local maps in (8.2) is compatible with the transition maps of $TM$ and $TM'$, and therefore defines a global derivative map

$$df\colon TM \longrightarrow TM'$$

satisfying the properties described in the first paragraph.

Moving to the derivation perspective of Section 7, a vector $v \in T_pM$ corresponds to a derivation

$$D\colon C^\infty_p(M,\mathbb{R}) \to \mathbb{R}.$$

To define $d_pf(v) \in T_{f(p)}M'$, we describe the corresponding derivation $D'$ acting on germs of smooth functions $h\colon U' \to \mathbb{R}$ at $f(p)$. Define

$$D'(h) = D(h\circ f) \qquad \forall\ h \in C^\infty_{f(p)}(M',\mathbb{R}).$$

It is straightforward to verify that $D'$ satisfies the Leibniz rule and is indeed a derivation. We leave it to the reader to verify that the two definitions are equivalent under the correspondence of Theorem 7.6.(i).

For every smooth map or holomorphic map $f\colon M \to N$, since

$$d_pf\colon T_pM \longrightarrow T_{f(p)}N$$

is a linear map, the quantity

$$\mathrm{rank}_p f := \mathrm{rank}\big(d_pf\colon T_pM \longrightarrow T_{f(p)}N\big) \le \dim M, \dim N$$

is well-defined and plays an important role in classifying different types of maps.

**Definition 8.1.** For every smooth or holomorphic map $f\colon M \to N$, we say:

- $f$ is an **immersion** if $d_pf$ is injective for all $p \in M$ (this requires $\dim M \le \dim N$);

- $f$ is a **submersion** if $d_pf$ is surjective for all $p \in M$ (this requires $\dim M \geq \dim N$);
- $f$ has **constant rank** if $\operatorname{rank}_p f = r$ for all $p \in M$ and some $r \geq 0$;
- $f$ is an **embedding** if $f$ is an immersion and a homeomorphism onto its image;

The first two are special cases of the third, corresponding to $r = \dim M$ and $r = \dim N$, respectively.

**Remark 8.2.** Note that $\operatorname{rank}_p f$ is upper semicontinuous in $p$, meaning that if $\operatorname{rank}_p f = r$, then $\operatorname{rank}_q f \geq r$ for all $q$ in a sufficiently small neighborhood of $p$. This follows from the fact that in local coordinates, the matrix of partial derivatives varies smoothly with $p$. If $\operatorname{rank}_p f = r$, there exists an $r \times r$ minor of the Jacobian matrix at $p$ with nonzero determinant. Since being nonzero is an open condition, that determinant remains nonzero in a neighborhood of $p$.

In particular, if the derivative is full rank at a point $p$, it remains full rank on a neighborhood of $p$:

(i) If $d_pf$ is injective at some $p \in M$, then $f$ is an immersion in a neighborhood of $p$.

(ii) If $d_pf$ is surjective at some $p \in M$, then $f$ is a submersion in a neighborhood of $p$.

**Definition 8.3.** We say $q \in N$ is a **regular value** if $d_pf$ is surjective for all $p \in f^{-1}(q)$. If $q$ is not a regular value, we say $q$ is a **critical** value.

Before discussing various cases and the importance of regular values, let us define the notion of submanifold that corresponds to embeddings (see Exercise 8.8.).

**Definition 8.4.** Given a manifold $N$ and a maximal atlas $\mathcal{A}$ of some regularity type, we say that a subset $M \subset N$ is a **submanifold** if for every point $p \in M$, there exists a chart $\varphi\colon U \to V \subset A$ around $p$ in $\mathcal{A}$ such that

$$\varphi(U \cap M) = V \cap A'$$

for some affine subspace $A' \subset A$.

**Lemma 8.5.** *(i) If $N$ is smooth or holomorphic and $M \subset N$ is a submanifold (with respect to the maximal atlas of the smooth or holomorphic structure), then $M$ inherits a smooth or holomorphic structure, respectively.*

*(ii) If $N_1, N_2$ are smooth or holomorphic, $M_i \subset N_i$ are submanifolds (with respect to the maximal atlases of the smooth or holomorphic structures), and $f\colon N_1 \longrightarrow N_2$ is smooth or holomorphic such that $f|_{M_1}$ maps into $M_2$, then the restriction $f|_{M_1}\colon M_1 \longrightarrow M_2$ is also smooth or holomorphic, respectively.*

**Proof. Part (i).** By definition, the charts defining the induced smooth or holomorphic structure on $M$ are the restrictions to $M$ of those charts $\varphi\colon U \longrightarrow V \subset A$ on $N$ for which $\varphi(U \cap M) = V \cap A'$ for some affine subspace $A' \subset A$. Given two such charts,

$$\varphi_i\colon U_i \longrightarrow V_i \subset A_i, \qquad i = 1, 2,$$

the transition map between the induced charts on $M$ is the restriction

$$\varphi_2 \circ \varphi_1^{-1}\big|_{A_1' \cap V_{1,2}}\colon V_{1,2} \cap A_1' \longrightarrow V_{2,1} \cap A_2'.$$

Since the map

$$\varphi_2 \circ \varphi_1^{-1}\colon V_{1,2} \longrightarrow V_{2,1}$$

is smooth or holomorphic by assumption, its restriction to any affine subspace is also smooth or holomorphic.

**Part (ii).** This part is similar: given charts

$$\varphi_i\colon U_i \longrightarrow V_i \subset A_i, \qquad i = 1, 2,$$

on $N_i$, the composition $\varphi_2 \circ f \circ \varphi_1^{-1}$ is smooth or holomorphic, and we are restricting it to an affine subspace of the domain. Such a restriction remains smooth or holomorphic. $\square$

**Remark 8.6.** Lemma 8.5 will be very useful in examples. For instance, in the solution to Exercise 3.8, the map $f$ is the restriction to $\mathbb{R} \times S^1$ of the smooth automorphism

$$\mathbb{R}^3 \to \mathbb{R}^3, \qquad (x, x_0, x_1) \mapsto \big(x, f(x)x_0, f(x)x_1\big).$$

Therefore, once we know that $\mathbb{R} \times S^1$ is a smooth submanifold of $\mathbb{R}^3$, one can immediately conclude that $f$ is smooth.

The result that allows us to characterize constant rank maps – especially immersions and submersions – is the Constant Rank Theorem stated below.

**Theorem 8.7** (Constant Rank Theorem)**.** *Suppose $f\colon M \longrightarrow N$ is a smooth or holomorphic map of constant rank $r$. Then, for every $p \in M$, there exist charts*

$$\varphi_1\colon U_1 \longrightarrow V_1 \subset A_1 \quad \text{around } p \text{ on } M$$

*and*

$$\varphi_2\colon U_2 \longrightarrow V_2 \subset A_2 \quad \text{around } f(p) \text{ on } N$$

*such that the map $\varphi_2 \circ f \circ \varphi_1^{-1}$ is the restriction of a linear map $L\colon A_1 \longrightarrow A_2$ of rank $r$.*

We will discuss the proof of the Constant Rank Theorem in Section 10 and explore some of its powerful applications throughout the remainder of this one.

**Exercise 8.8.** Use the Constant Rank Theorem to show that $M \subset N$ is a smooth submanifold if and only if it is the image of an embedding.

**Lemma 8.9.** *Suppose $f\colon M \longrightarrow N$ is a smooth or holomorphic map and $q \in N$ is a regular value. Then the* **level set** $Y = f^{-1}(q)$ *is a smooth or holomorphic submanifold of $M$, respectively. Furthermore,* $\dim Y = \dim M - \dim N$ *and*

$$T_pY = \ker(d_pf\colon T_pM \longrightarrow T_{f(p)}N) \qquad \forall\ p \in Y.$$

**Proof.** By Remark 8.2, $f$ is a submersion on an open neighborhood $U$ of $Y$. Since $\mathrm{rank}_p f = \dim N$ for all $p \in U$, the Constant Rank Theorem (applied to $f|_U$) implies that for every $p \in Y$, there exist charts

$$\varphi_1\colon U_1 \longrightarrow V_1 \subset A_1 \quad \text{around } p, \qquad \varphi_2\colon U_2 \longrightarrow V_2 \subset A_2 \quad \text{around } q,$$

with $\varphi_1(p) = 0$ and $\varphi_2(q) = 0$, such that the map $\varphi_2 \circ f \circ \varphi_1^{-1}$ is the restriction of a surjective linear map $L\colon A_1 \longrightarrow A_2$. Therefore,

$$\varphi_1(Y \cap U_1) = \ker(L) \cap V_1.$$

This shows that every point $p \in Y$ admits a chart as in Definition 8.4, with $A' = \ker(L)$. The last two observations also follow from the fact that the induced chart on $Y$ takes values in the affine subspace $\ker(L)$. $\square$

Theorem 8.8 is a powerful tool for constructing interesting manifolds as submanifolds of simpler ones like $\mathbb{R}^m$, fulfilling item 4 in the introductory paragraph of Section 5. Here's an example, with many more to be discussed in the exercises.

**Example 8.10.** The $m$-dimensional unit sphere $S^m$ arises as the level set

$$S^m = f^{-1}(1) \subset \mathbb{R}^{m+1}$$

of the smooth function

$$f\colon \mathbb{R}^{m+1} \longrightarrow \mathbb{R}, \qquad x = (x_0, \dots, x_m) \mapsto \sum_{i=0}^{m} x_i^2.$$

Its differential is given by

$$df = 2\sum_{i=0}^{m} x_i\, dx_i,$$

where for each $x \in \mathbb{R}^{m+1}$, the map $dx_i\colon T_x\mathbb{R}^{m+1} \longrightarrow T_{f(x)}\mathbb{R} = \mathbb{R}$ is the linear function that sends $\partial_{x_i}$ to 1 and $\partial_{x_j}$ to 0 for all $j \neq i$.

In general, for a smooth function $f\colon M \to \mathbb{R}$, a point $q \in \mathbb{R}$ is a regular value if and only if $d_pf \neq 0$ for all $p \in f^{-1}(q)$. In this example, the only point where $df$ vanishes is the origin, which does not lie on $S^m$. We conclude that $S^m$ is a regular level set and inherits a smooth manifold structure from $\mathbb{R}^{m+1}$.

It is easy to check that this agrees with the smooth structure described in Section 2.

**Exercise 8.11.** Show that the matrix groups $\mathrm{O}(n)$, $\mathrm{SU}(n)$, and $\mathrm{SL}(n,\mathbb{R})$ are smooth manifolds. What is the dimension of each? Show that $\mathrm{SU}(2)$ is diffeomorphic to $S^3$.

**Exercise 8.12.** Provide a smooth embedding of $S^m \times S^n$ into $\mathbb{R}^{m+n+1}$.

**Exercise 8.13.** Show that the map

$$f\colon \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}^5, \quad f(x,y,z) = (xy, yz, zx, x^2 - y^2, x^2 + y^2 + z^2 - 1)$$

is an immersion. Use $f$ to construct an embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{R}^4$.

**Exercise 8.14.** For $a > b > 0$, show that the surface

$$M = \{(x,y,z) \in \mathbb{R}^3 \mid (r-a)^2 + z^2 = b^2\}$$

is diffeomorphic to a 2-torus. Here, $r^2 = x^2 + y^2$.

**Exercise 8.15.** For every $4 \times 2$ matrix $A$ and $1 \le i < j \le 4$, let $A_{ij}$ denote the $2 \times 2$ minor of $A$ corresponding to the $i$-th and $j$-th rows, and define

$$f(A) = \big(\det(A_{12}), \ldots, \det(A_{34})\big) \in \mathbb{R}^6.$$

Use $f$ to construct an embedding (called the Plücker embedding) of the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$ into $\mathbb{RP}^5$.

**Exercise 8.16.** Over $\mathbb{R}$ or $\mathbb{C}$, we define an elliptic curve $C$ to be the solution set of a cubic equation

$$\big(y^2 = x^3 + ax + b\big) \subset \mathbb{R}^2 \text{ or } \mathbb{C}^2,$$

where $a$ and $b$ are real or complex, depending on the context. Find conditions on $P$ under which $C$ is a smooth manifold. In the complex case, determine whether the closure of $C$ in $\mathbb{CP}^2$ is also a (compact) manifold.

## Solutions to exercises

**Exercise 8.8.** If $M \subset N$ is a smooth submanifold, then the inclusion map $\iota\colon M \to N$ is an embedding.

Conversely, suppose $M$ is the image of an embedding $f\colon M' \longrightarrow N$. By the Constant Rank Theorem, for every $p \in M'$, there exist charts

$$\varphi_1\colon U_1 \longrightarrow V_1 \subset A_1 \quad \text{around } p \text{ on } M',$$

and

$$\varphi_2\colon U_2 \longrightarrow V_2 \subset A_2 \quad \text{around } f(p) \text{ on } N,$$

such that the map $\varphi_2 \circ f \circ \varphi_1^{-1}$ is the restriction of an injective linear map $L\colon A_1 \longrightarrow A_2$, and it is a homeomorphism onto its image.

Let $A' = L(A_1)$. The fact that $f$ is a homeomorphism onto its image implies that

$$\varphi_2(M \cap U_2) = V_2 \cap A',$$

i.e., every point $f(p) \in M$ admits a chart as in Definition 8.4. $\square$

**Exercise 8.11.** We have $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$, where

$$\det\colon M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$$

is the determinant function. Since det is a polynomial in the matrix entries, it is smooth. To show that 1 is a regular value, it suffices to show that $d_A \det \neq 0$ for all $A \in \mathrm{SL}(n,\mathbb{R})$.

Consider the curve $\gamma(t) = e^t A$ through $A$. Then,

$$\frac{d}{dt}\det(e^t A) = \frac{d}{dt}\big(e^{nt}\det(A)\big) = ne^{nt}\det(A).$$

At $t = 0$, this derivative is $n \cdot \det(A) = n \neq 0$, since $\det(A) = 1$. Therefore, $d_A \det \neq 0$ for all $A \in \mathrm{SL}(n,\mathbb{R})$, and so 1 is a regular value.

We have

$$\mathrm{O}(n) = \Big\{A \in M_{n\times n}(\mathbb{R})\colon A^T A = \mathrm{I}_n\Big\}.$$

Thus, $\mathrm{O}(n) = f^{-1}(\mathrm{I}_n)$, where

$$f\colon M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2} \longrightarrow M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n(n+1)/2}$$

is the map $A \mapsto A^T A$ into the space of symmetric $n \times n$ matrices. By the product rule, the derivative

$$d_A f\colon T_A M_{n\times n}(\mathbb{R}) = M_{n\times n}(\mathbb{R}) \longrightarrow T_{f(A)} M^{\mathrm{sym}}_{n\times n}(\mathbb{R}) = M^{\mathrm{sym}}_{n\times n}(\mathbb{R})$$

is given by

$$d_A f(B) = A^T B + B^T A = (A^T B) + (A^T B)^T.$$

Given $A \in \mathrm{O}(n)$ and any symmetric matrix $C$, let

$$B = \frac{1}{2} AC.$$

Since $A^{-1} = A^T$, we have $d_A f(B) = C$. Therefore, $d_A f$ is surjective for all $A \in \mathrm{O}(n)$, so $\mathrm{I}_n$ is a regular value.

We have

$$\mathrm{SU}(n) = \Big\{ A \in \mathrm{U}(n) \colon \det A = 1 \Big\}, \quad \mathrm{U}(n) = \Big\{ A \in M_{n\times n}(\mathbb{C}) \colon A^\dagger A = \mathrm{I}_n \Big\},$$

where $A^\dagger = \overline{A}^T$ is the Hermitian transpose. First, $\mathrm{U}(n)$ is the preimage

$$\mathrm{U}(n) = f^{-1}(\mathrm{I}_n)$$

under the smooth map

$$f \colon M_{n\times n}(\mathbb{C}) \longrightarrow \mathrm{Herm}(n), \qquad A \mapsto A^\dagger A,$$

where $\mathrm{Herm}(n) \subset M_{n\times n}(\mathbb{C})$ denotes the real vector space of Hermitian matrices. The proof that $\mathrm{I}_n$ is a regular value of $f$ is very similar to the case of $\mathrm{O}(n)$.

Furthermore, since $\det \colon M_{n\times n}(\mathbb{C}) \longrightarrow \mathbb{C}$ is smooth, it restricts to a smooth map $\det \colon \mathrm{U}(n) \longrightarrow S^1 \subset \mathbb{C}$. We aim to show that 1 is a regular value of the latter, i.e., $d_A \det \neq 0$ for all $A \in \mathrm{U}(n)$.

By Jacobi's formula for the derivative of the determinant, we have

$$d_A \det(B) = \det(A) \operatorname{tr}(A^{-1}B) \qquad \forall\ A \in \mathrm{U}(n).$$

By the second statement in Lemma 8.9,

$$T_A \mathrm{U}(n) = \big\{ B \in M_{n\times n}(\mathbb{C}) \colon A^\dagger B + (A^\dagger B)^\dagger = 0 \big\}.$$

Choose a skew-Hermitian matrix $D$ such that $\operatorname{tr}(D) = \mathfrak{i}$, and let $B = AD$. Then, for $A \in \mathrm{SU}(n)$,

$$d_A \det(B) = \mathfrak{i} \in T_1 S^1 \cong \mathbb{R} \cdot \mathfrak{i} \subset \mathbb{C}.$$

Therefore, 1 is a regular value of the determinant function on $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ is a smooth real codimension-one submanifold of $\mathrm{U}(n)$.

**Remark 8.17.** One can obtain $\mathrm{SU}(n)$ as a level set in one shot by considering the map

$$f \colon M_{n\times n}(\mathbb{C}) \longrightarrow \mathrm{Herm}(n) \times \mathbb{C}, \qquad A \mapsto \big(A^\dagger A, \det(A)\big).$$

However, $(\mathrm{I}_n, 1)$ is not a regular value, since $f$ is a constant rank map of rank

$$r = \dim_{\mathbb{R}}(\mathrm{Herm}(n) \times \mathbb{C}) - 1.$$

The action of $\mathrm{SU}(2)$ on $\mathbb{C}^2$ preserves the unit sphere $S^3 \subset \mathbb{C}^2$. For every $(a,b) \in S^3 \subset \mathbb{C}^2$, the matrix

$$A = \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix}$$

lies in $\mathrm{SU}(2)$, and its action on $S^3$ sends the point $(1,0)$ to $(a,b)$. Thus, the action of $\mathrm{SU}(2)$ on $S^3$ is transitive.

It remains to show that the stabilizer (isotropy group) of this action is trivial. By transitivity, it suffices to check the stabilizer at the point $(1,0)$. Suppose

$$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SU}(2).$$

Then $a = 1$ and $c = 0$. Since $A$ is unitary with determinant 1, these conditions imply $b = 0$ and $d = 1$, so $A = \mathrm{I}_2$.

We conclude that the action is free and transitive, so $\mathrm{SU}(2) \cong S^3$ as smooth manifolds. More precisely, every matrix in $\mathrm{SU}(2)$ is of the form

$$A = \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix}$$

with $(a,b) \in S^3$. $\square$

**Exercise 8.12.** With the standard embeddings

$$\iota_1 \colon S^m \longrightarrow \mathbb{R}^{m+1} \quad \text{and} \quad \iota_2 \colon S^n \longrightarrow \mathbb{R}^{n+1},$$

the product embedding

$$\iota_1 \times \iota_2 \colon S^m \times S^n \longrightarrow \mathbb{R}^{m+n+2}$$

lands in the sphere $S^{m+n+1}(\sqrt{2}) \subset \mathbb{R}^{m+n+2}$ of radius $\sqrt{2}$. However, the image of $\iota_1 \times \iota_2$ does not cover the whole sphere; for instance, it misses the point $p = (\sqrt{2}, 0, \ldots, 0)$.

Stereographic projection from $p$ identifies $S^{m+n+1}(\sqrt{2}) \setminus \{p\}$ with $\mathbb{R}^{m+n+1}$. Composing this projection with the product embedding gives the desired embedding of $S^m \times S^n$ into $\mathbb{R}^{m+n+1}$. $\square$

**Exercise 8.13.**

We have

$$d_{(x,y,z)}f = \begin{bmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \\ 2x & -2y & 0 \\ 2x & 2y & 2z \end{bmatrix}.$$

If $xyz \neq 0$, then the top $3 \times 3$ minor is invertible, so $d_{(x,y,z)}f$ is full rank.

If $x = 0$, then

$$d_{(0,y,z)}f = \begin{bmatrix} y & 0 & 0 \\ 0 & z & y \\ z & 0 & 0 \\ 0 & -2y & 0 \\ 0 & 2y & 2z \end{bmatrix}.$$

If $y \neq 0$, the minor

$$\begin{bmatrix} y & 0 & 0 \\ 0 & z & y \\ 0 & -2y & 0 \end{bmatrix}$$

is full rank. If $y = 0$ and $z \neq 0$, the minor

$$\begin{bmatrix} 0 & z & 0 \\ z & 0 & 0 \\ 0 & 0 & 2z \end{bmatrix}$$

is full rank.

The case $d_{(x,0,z)}f$ is similar by symmetry. Finally, we consider

$$d_{(x,y,0)}f = \begin{bmatrix} y & x & 0 \\ 0 & 0 & y \\ 0 & 0 & x \\ 2x & -2y & 0 \\ 2x & 2y & 0 \end{bmatrix},$$

and we may assume $x, y \neq 0$, in which case the minor

$$\begin{bmatrix} 0 & 0 & x \\ 2x & -2y & 0 \\ 2x & 2y & 0 \end{bmatrix}$$

is full rank. We conclude that $f$ is an immersion. Since $f$ is defined on $\mathbb{R}^3$ and $f|_{S^2}$ maps into $\mathbb{R}^4$ (because $x^2 + y^2 + z^2 - 1 = 0$), the restriction

$$f|_{S^2} \colon S^2 \longrightarrow \mathbb{R}^4, \quad (x, y, z) \mapsto (xy, yz, zx, x^2 - y^2)$$

is also an immersion.

Note that $f(x, y, z) = f(-x, -y, -z)$, and since immersion is a local property, $f|_{S^2}$ descends to an immersion

$$\overline{f} \colon \mathbb{RP}^2 = S^2/\mathbb{Z}_2 \longrightarrow \mathbb{R}^4.$$

For compact manifolds $M$, to show that $f \colon M \to N$ is an embedding, it suffices to prove that it is a one-to-one immersion. Thus, it remains to show that

$$f|_{S^2} \colon S^2 \to \mathbb{R}^4$$

is 2-to-1 in order to conclude that $\overline{f} \colon \mathbb{RP}^2 \to \mathbb{R}^4$ is one-to-one.

Suppose
$$(xy, yz, zx, x^2 - y^2) = (a, b, c, d),$$
and assume $abc \neq 0$. Then we can solve:
$$x^2 = \frac{ac}{b}, \quad y^2 = \frac{ab}{c}, \quad z^2 = \frac{bc}{a}.$$
So $(x, y, z)$ is determined by $(a, b, c)$ up to an overall sign. Choosing one of the two possible signs for $x$ uniquely determines $y$ and $z$.

If exactly one of $x, y, z$ is zero, then exactly two of $a, b, c$ are zero. If exactly two of $x, y, z$ are zero, then all of $a, b, c$ are zero. In each case, fixing the sign of one nonzero coordinate determines the others uniquely. Thus, $f|_{S^2}$ is 2-to-1, and $\overline{f}$ is an embedding. □

**Exercise 8.14.** The equation is written in cylindrical coordinates $(r, \theta, z)$ and is independent of $\theta$. In each fixed $\theta$-plane, the equation $(r-a)^2+z^2 = b^2$ describes a circle of radius $b$. Therefore, the surface $M$ is diffeomorphic to $S^1 \times S^1$.

More precisely, in Euclidean coordinates, the map
$$h\colon S^1\times S^1 \longrightarrow \mathbb{R}^3, \quad (\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\varphi}) \longmapsto \Big((a+b\cos\varphi)\cos\theta,\ (a+b\cos\varphi)\sin\theta,\ b\sin\varphi\Big)$$
is a one-to-one immersion (hence an embedding) into $\mathbb{R}^3$, whose image is the surface $M$.

□

**Exercise 8.15.** Every plane $V \subset \mathbb{R}^4$ is the span of two linearly independent (column) vectors $v_1, v_2$. Putting them together, we obtain a $4 \times 2$ matrix $A = A(v_1, v_2)$. Different bases $(v_1, v_2)$ of $V$ are related by the right action of $\mathrm{GL}(2, \mathbb{R})$ on $A$. Therefore,
$$\mathrm{Gr}_2(\mathbb{R}^4) = \{A \in M_{4\times 2}(\mathbb{R})\colon \operatorname{rank}(A) = 2\}/\mathrm{GL}(2, \mathbb{R}).$$
The function $f\colon M_{4\times 2}(\mathbb{R}) \to \mathbb{R}^6$ descends to a well-defined function
$$\begin{aligned}&\overline{f}\colon \mathrm{Gr}_2(\mathbb{R}^4) \longrightarrow \mathbb{RP}^5,\\ &[A] \longmapsto \big[\det(A_{12}) : \det(A_{13}) : \det(A_{14}) : \det(A_{23}) : \det(A_{24}) : \det(A_{34})\big],\end{aligned}$$
where $[A]$ denotes the class of $A$ in the quotient.

We aim to show that $\overline{f}$ is an embedding, i.e., a one-to-one immersion. For every $A$, there exist indices $i < j$ such that $\det(A_{ij}) \neq 0$. Therefore, there is a unique representative of $[A]$ such that $A_{ij} = \mathrm{I}_2$. By symmetry, we may assume $(i, j) = (1, 2)$; the other cases are similar. If $A_{12} = \mathrm{I}_2$, then
$$A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix},$$

and

$$\overline{f}([A]) = \big[1 : c : d : -a : -b : ad - bc\big].$$

In the terminology of (2.8) and (2.11):

- $[A]$ belongs to the domain of the chart $\varphi_{12}\colon U_{12} \to V_{12} \cong M_{2\times 2}(\mathbb{R})$ on $\mathrm{Gr}_2(\mathbb{R}^4)$;
- $\overline{f}([A])$ lies in the domain of the chart $\varphi_1\colon U_1 \to V_1$ on $\mathbb{RP}^5$;
- and the composition $\varphi_1 \circ \overline{f} \circ \varphi_{12}^{-1}\colon M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^5$ is the smooth embedding

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (c, d, a, b, ad - bc).$$

Since $\overline{f}^{-1}(U_1)$ is precisely $U_{12}$, we conclude that $\overline{f}$ is a smooth embedding.

**Remark 8.18.** The construction above also works over $\mathbb{C}$ and generalizes to define embeddings

$$\mathrm{Gr}_k(\mathbb{R}^n) \longrightarrow \mathbb{RP}^{\binom{n}{k}-1} \quad \text{and} \quad \mathrm{Gr}_k(\mathbb{C}^n) \longrightarrow \mathbb{CP}^{\binom{n}{k}-1}.$$

$\square$

**Exercise 8.16.** The real or complex curve $C$ is the 0-level set of the function $P(x, y) = y^2 - (x^3 + ax + b)$. We have

$$dP = 2y\, dy - (3x^2 + a)\, dx.$$

Therefore, 0 is a critical value if and only if there is $x_0$ such that

$$3x_0^2 + a = 0 \quad \text{and} \quad x_0^3 + ax_0 + b = 0;$$

in other words, $x_0$ must be a double (or triple) root of $x^3 + ax + b$. Numerically, the equations above imply

$$x_0^2 = -\frac{a}{3} \Rightarrow x_0 \left(\frac{2a}{3}\right) + b = 0 \Rightarrow$$

$$x_0 = -\frac{3b}{2a} \Rightarrow 3\left(-\frac{3b}{2a}\right)^2 = -a \Rightarrow 27b^2 + 4a^3 = 0.$$

Conversely, if $27b^2 + 4a^3 = 0$, it is easy to check that $x_0 = -\frac{3b}{2a}$ is a double (or triple) root. (If $a = b = 0$, we get a triple root at $x_0 = 0$).

If $x_0$ is a double root, then the equation has the form $y^2 = (x - x_0)^2(x - \lambda)$ for some $\lambda \neq x_0$. Therefore,

- if we are working over $\mathbb{R}$ and $\lambda < x_0$, then near $x = x_0$, $C$ has two local intersecting branches $y = \pm(x - x_0)\sqrt{x - \lambda}$; i.e., $C$ is singular (not a manifold).

- if we are working over $\mathbb{R}$ and $\lambda > x_0$, then $(x_0, 0)$ is an isolated point of $C$, i.e., $C$ is degenerate at that point (not a 1-dimensional manifold).
- if we are working over $\mathbb{R}$ and $x_0$ is a triple root, then $C$ has a cusp singularity at $(x_0, 0)$ (again, not a manifold).
- if we are working over $\mathbb{C}$ and $x_0$ is a double root, then again, $C$ has two local intersecting branches.
- if we are working over $\mathbb{C}$ and $x_0$ is a triple root, then $C$ has a cusp singularity at $(x_0, 0)$.

Recall that $\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1$, where $\mathbb{C}^2$ is the set of points $[1 : x_1 : x_2]$ and $\mathbb{CP}^1$ is the set of points $[0 : x_1 : x_2]$. In other words, $\mathbb{CP}^2$ can be covered by three charts

$$\varphi_i \colon U_i \to V_i, \quad \varphi_i\big([X_0 : X_1 : X_2]\big) = \left(x_j = \frac{X_j}{X_i}\right)_{j \neq i}, \quad \text{for } i = 0, 1, 2,$$

where we think of $U_0$ as $\mathbb{C}^2$, and $\big(U_1 \cup U_2\big) \setminus U_0$ as the $\mathbb{CP}^1$ added to compactify it. In order to understand the closure $\overline{C}$ of $C$ in $\mathbb{CP}^2$, we need to understand its equation in terms of the chart variables on $U_1$ and $U_2$ to find the extra solution points that do not belong to $U_0$. We then study whether $\overline{C}$ is a manifold near those points.

To visually distinguish the coordinates on $V_0$, $V_1$, and $V_2$, let

- $x_1 = \frac{X_1}{X_0}$ and $x_2 = \frac{X_2}{X_0}$ denote the coordinates on $V_0$;
- $y_0 = \frac{X_0}{X_1}$ and $y_2 = \frac{X_2}{X_1}$ denote the coordinates on $V_1$;
- $z_0 = \frac{X_0}{X_2}$ and $z_1 = \frac{X_1}{X_2}$ denote the coordinates on $V_2$;

Switching the notation from $(x, y)$ to $(x_1, x_2)$ for $C$, we are starting from the equation

$$x_2^2 - (x_1^3 + ax_1 + b) = 0 \tag{8.3}$$

on $V_0$, and we want to find its equivalent on $V_1$ and $V_2$.

The coordinates $(x_1, x_2)$ and $(y_0, y_2)$ are related by

$$x_1 = \frac{1}{y_0}, \quad x_2 = \frac{y_2}{y_0}.$$

Substituting these into (8.3) and multiplying by $y_0^3$ gives

$$y_0 y_2^2 - (1 + ay_0^2 + by_0^3) = 0.$$

The points in $\overline{C} \cap (U_1 \setminus U_0)$ correspond to setting $y_0 = 0$, which yields no solution.

Similarly, the coordinates $(x_1, x_2)$ and $(z_0, z_1)$ are related by

$$x_1 = \frac{z_1}{z_0}, \quad x_2 = \frac{1}{z_0}.$$

Substituting these into (8.3) and multiplying by $z_0^3$ gives

$$z_0 - (z_1^3 + az_1z_0^2 + bz_0^3) = 0.$$

The points in $\overline{C} \cap (U_2 \setminus U_0)$ correspond to setting $z_0 = 0$, which yields the solution $(z_0, z_1) = (0, 0)$. Therefore,

$$\overline{C} = C \cup \{[0 : 0 : 1]\}.$$

Let us now understand the behavior of $\overline{C}$ near $[0 : 0 : 1]$. Differentiating the equation above in $z$-coordinates and evaluating at $(0, 0)$ gives

$$(1 - 2az_1z_0)\, dz_0 + (3z_1^2 + az_0^2)\, dz_1\big|_{(0,0)} = dz_0 \neq 0.$$

Therefore, $\overline{C}$ is non-singular at $[0 : 0 : 1]$.

**Conclusion.** If $a, b \in \mathbb{C}$ and $27b^2 + 4a^3 \neq 0$, then $\overline{C} \subset \mathbb{CP}^2$ is a holomorphic manifold of complex dimension one – that is, a closed Riemann surface when viewed as a real manifold. Determining the genus of this surface requires tools that will be introduced in the next book.

$\square$

Chapter 9

# Transversality

In this lecture, we gradually generalize our previous results on regular values and regular level sets to their most general form. First, we consider the preimage of a larger submanifold than just a point.

**Definition 9.1.** Suppose $f\colon M \longrightarrow N$ is a smooth map and $Z \subset N$ is a smooth submanifold. We say that $f$ is **transverse** to $Z$ if

$$d_pf(T_pM) + T_{f(p)}Z = T_{f(p)}N \qquad \forall\, p \in f^{-1}(Z).$$

In particular, we say that two submanifolds $Z_1, Z_2 \subset N$ are transverse, and write $Z_1 \pitchfork Z_2$, if the inclusion map $\iota_1\colon Z_1 \longrightarrow N$ is transverse to $Z_2$. In other words,

$$T_pZ_1 + T_pZ_2 = T_pN \qquad \forall\, p \in Z_1 \cap Z_2.$$

**Proposition 9.2.** *Suppose $f\colon M \longrightarrow N$ is a smooth or holomorphic map, and $Z \subset N$ is a submanifold of the corresponding regularity class. If $f$ is transverse to $Z$, then the preimage $Y = f^{-1}(Z)$ is a smooth or holomorphic submanifold of $M$, respectively. Moreover,* $\operatorname{codim}_M Y = \operatorname{codim}_N Z$, *and*

$$T_pY = d_pf^{-1}(T_{f(p)}Z) = \ker\big(d_pf\colon T_pM \longrightarrow T_{f(p)}N/T_{f(p)}Z\big) \qquad \forall\, p \in Y.$$

*In particular, if $Z_1, Z_2 \subset N$ are transverse submanifolds, then $Z_1 \cap Z_2$ is also a submanifold of codimension* $\operatorname{codim}_N Z_1 + \operatorname{codim}_N Z_2$.

**Proof.** For each $p \in Y$, choose an open neighborhood $W$ of $p$ in $M$ and a chart $\varphi\colon U \longrightarrow V \subset A$ around $f(p)$ in $N$ such that

- $\varphi(U \cap Z) = V \cap A'$ for some affine subspace $A' \subset A$ as in Definition 8.4, and
- $f(W) \subset U$.

Then,

$$(\pi \circ \varphi \circ f)^{-1}(0) = Y \cap W,$$

where $\pi\colon A \longrightarrow A/A'$ is the quotient projection map.

By the transversality assumption, 0 is a regular value of the map

$$\pi \circ \varphi \circ f\colon W \longrightarrow A/A'.$$

It then follows from Lemma 8.9 that $Y \cap W$ is a smooth or holomorphic submanifold of the expected dimension. Since this holds in a neighborhood of every $p \in Y$, we conclude that $Y$ is a smooth or holomorphic submanifold of $M$ of the expected codimension. □

Next we discuss the full generalization of Lemma 8.9 leading to the concept of fiber product.

**Definition 9.3.** For $i = 1, 2$, suppose $f_i\colon M_i \longrightarrow N$ are smooth maps. We say that $f_1$ is transverse to $f_2$, and write $f_1 \pitchfork f_2$, if

$$\begin{aligned} &d_{p_1} f_1(T_{p_1} M_1) + d_{p_2} f_2(T_{p_2} M_2) = T_q N, \\ &\forall\ (p_1, p_2) \in M_1 \times M_2 \quad \text{with } f_1(p_1) = f_2(p_2) = q \in N. \end{aligned}$$

**Theorem 9.4.** *For $i = 1, 2$, suppose $f_i\colon M_i \longrightarrow N$ are smooth or holomorphic maps, and assume $f_1 \pitchfork f_2$. Then the* **fiber product**

$$M_1\ {}_{f_1}\!\times_{f_2} M_2 := \big\{(p_1, p_2) \in M_1 \times M_2\colon f_1(p_1) = f_2(p_2)\big\} \subset M_1 \times M_2$$

*is a smooth or holomorphic submanifold of $M_1 \times M_2$, respectively. Moreover,*

$$\dim\big(M_1\ {}_{f_1}\!\times_{f_2} M_2\big) = \dim M_1 + \dim M_2 - \dim N,$$

*and*

$$T_{(p_1,p_2)}\big(M_1\ {}_{f_1}\!\times_{f_2} M_2\big) = \big\{(v_1, v_2) \in T_{p_1} M_1 \times T_{p_2} M_2\colon d_{p_1} f_1(v_1) = d_{p_2} f_2(v_2)\big\}.$$

**Proof.** It is clear that $M_1\ {}_{f_1}\!\times_{f_2} M_2$ is the preimage under $f_1 \times f_2$ of the diagonal subset $N \cong \Delta_N \subset N \times N$. Given any atlas $\mathcal{A} = \{\varphi_\alpha\colon U_\alpha \longrightarrow V_\alpha \subset A_\alpha\}_{\alpha \in \mathcal{I}}$, we obtain the product atlas

$$\mathcal{A} \times \mathcal{A} := \{\varphi_\alpha \times \varphi_\beta\colon U_\alpha \times U_\beta \longrightarrow V_\alpha \times V_\beta\}_{\alpha,\beta \in \mathcal{I}}$$

which defines the product smooth or holomorphic structure on $N \times N$. For every $\alpha \in \mathcal{I}$, we have

$$\varphi_\alpha \times \varphi_\alpha\big((U_\alpha \times U_\alpha) \cap \Delta_N\big) = \big(V_\alpha \times V_\alpha\big) \cap \Delta_{A_\alpha},$$

where $\Delta_{A_\alpha}$ is the diagonal subspace in $A_\alpha \times A_\alpha$. Therefore, $\Delta_N \subset N \times N$ is a smooth or holomorphic submanifold, depending on the context.

Note that

$$T_{(q,q)}\Delta_N = \{(w, w) \in T_{(q,q)}(N \times N)\colon w \in T_q N\}.$$

**Claim.** The product map $f_1 \times f_2$ is transverse to $\Delta_N$ if and only if $f_1$ is transverse to $f_2$. Therefore, the theorem follows from Proposition 9.2.

**Proof of Claim.** Suppose $f_1$ is transverse to $f_2$, i.e.,

$$d_{p_1} f_1(T_{p_1} M_1) + d_{p_2} f_2(T_{p_2} M_2) = T_q N,$$
$$\forall\ (p_1, p_2) \in M_1 \times M_2 \quad \text{with } f_1(p_1) = f_2(p_2) = q \in N.$$

We aim to show that $f_1 \times f_2$ is transverse to $\Delta_N$. For any

$$(w_1, w_2) \in T_{(q,q)}(N \times N) \cong T_q N \oplus T_q N,$$

by assumption, there exist

$$v_1 \in T_{p_1} M_1 \quad \text{and} \quad v_2 \in T_{p_2} M_2$$

such that

$$d_{p_1} f_1(v_1) + d_{p_2} f_2(-v_2) = w_1 - w_2.$$

Define

$$u := w_1 - d_{p_1} f_1(v_1) = w_2 - d_{p_2} f_2(v_2) \in T_q N.$$

Then

$$d_{(p_1,p_2)}(f_1 \times f_2)(v_1, v_2) + (u, u) = (w_1, w_2).$$

Therefore,

$$d_{(p_1,p_2)}(f_1 \times f_2)(T_{(p_1,p_2)}(M_1 \times M_2)) + T_{(q,q)}\Delta_N = T_{(q,q)}(N \times N),$$
$$\forall\ q \in N,\ (p_1, p_2) \in (f_1, f_2)^{-1}(q, q);$$

i.e., $f_1 \times f_2$ is transverse to $\Delta_N$. The converse follows similarly. $\square$ $\square$

**Example 9.5.** Here is an important example of the concepts above that will eventually lead to the definition of the Euler characteristic, once we introduce the notion of orientation on manifolds.

For every smooth or holomorphic manifold $M$, its tangent bundle, considered as a smooth or holomorphic manifold of twice the dimension, contains a canonical copy of $M$ as the zero-section. We denote this zero-section by $M_0 \subset TM$. In other words, $M$ is embedded into $TM$ via the map $x \mapsto 0_x \in T_x M$, and $M_0$ denotes the image of this embedding.

More generally, any section of the tangent bundle – that is, any vector field $\xi\colon M \to TM$ – defines an embedding of $M$ into $TM$. That, this is a smooth or holomorphic embedding follows from differentiating $\pi \circ \xi = \mathrm{id}$. We say that $\xi$ is a **transverse section** (or **transverse vector field**) if it is transverse to the zero-section $M_0 \subset TM$, in the sense of Definition 9.1. In this case, the set of points at which $\xi$ vanishes, namely $\xi^{-1}(M_0)$, is a submanifold of codimension

$$\operatorname{codim}_{TM}(M_0) = \dim M.$$

That is, $\xi^{-1}(M_0)$, often simply denoted $\xi^{-1}(0)$, is a discrete set of isolated points (finite if $M$ is compact).

In local coordinates, the condition of transversality for $\xi$ has an explicit form. Suppose $(x_1, \dots, x_m)$ are local coordinates on an open subset $U \subset M$. Then $\xi$ takes the form

$$\xi(x) = (x, y(x)) \in T\mathbb{R}^m \cong \mathbb{R}^m \times \mathbb{R}^m,$$

meaning that

$$\xi(x) = \big(x, \sum_{i=1}^m y_i(x)\, \partial_{x_i}\big).$$

Therefore, the derivative $d\xi$ is the matrix

$$d\xi = \begin{bmatrix} \mathrm{I}_m \\ \left[\frac{\partial y_j}{\partial x_i}\right] \end{bmatrix}.$$

Since the tangent space to the zero-section $TM_0$ corresponds to the span of the columns of

$$\begin{bmatrix} \mathrm{I}_m \\ 0_m \end{bmatrix},$$

we conclude that a point $p \in \xi^{-1}(0)$ is a transverse zero if and only if the matrix $\left[\frac{\partial y_j}{\partial x_i}(p)\right]$ of partial derivatives of the coefficients of $\xi$ at $p$ is non-singular.

**Exercise 9.6.** Recall that the 2-sphere $S^2 \subset \mathbb{R}^3$ can be covered by two charts $\varphi_\pm \colon U_\pm \to V_\pm \cong \mathbb{R}^2$, with transition map

$$\varphi_{+\mapsto-} \colon V_{+,-} = \mathbb{R}^2 \backslash \{0\} \to V_{-,+} = \mathbb{R}^2 \backslash \{0\}, \quad x = (x_1, x_2) \mapsto (y_1, y_2) = \frac{1}{|x|^2}(x_1, x_2).$$

In the solution to Exercise 6.10, we showed that the local vector fields

$$\xi_+ = x_1 \partial_{x_1} + x_2 \partial_{x_2} \quad \text{on } V_+$$

and

$$\xi_- = -(y_1 \partial_{y_1} + y_2 \partial_{y_2}) \quad \text{on } V_-$$

are compatible on the overlap and define a global vector field $\xi$ on $S^2$. Is this vector field transverse to the zero section?

**Exercise 9.7.** In the solution to Exercise 6.12, we showed that every holomorphic vector field $\xi$ on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is the extension to $\infty$ of a vector field

$$\xi_0 = (a + bz + cz^2)\partial_z$$

on $\mathbb{C}$. For which $a, b, c \in \mathbb{C}$ is $\xi$ transverse to the zero section? How many zeros does it have?

**Exercise 9.8.** Suppose $f\colon M \to M$ is an automorphism of the smooth manifold $M$. The graph $M_f$ of $f$ is the image of the embedding

$$M \to M \times M, \qquad p \mapsto (p, f(p)) \quad \forall\, p \in M.$$

For instance, the diagonal $\Delta_M$ is the graph of the identity map. Show that $M_f$ and $\Delta_M$ are transverse if and only if, for every fixed point $p$ of $f$, the differential $d_p f\colon T_pM \to T_pM$ has no eigenvalue equal to 1. Note that if $M_f$ and $\Delta_M$ are transverse, the fixed points of $f$ will be isolated, since $\dim(\Delta_M \cap M_f) = 0$.

**Exercise 9.9.** Let $f\colon \mathbb{C} \to \mathbb{C}^2$ be the holomorphic map

$$z \mapsto (z^a, z^b)$$

for some positive integers $a$ and $b$ with $\gcd(a, b) = 1$. Show that the smooth map $f$ is transverse to $S^3 \subset \mathbb{C}^2$. What is the intersection of the image of $f$ with $S^3$?

**Exercise 9.10.** Show that the following subsets of $\mathbb{CP}^2$ are holomorphic submanifolds:

$$Q_1 = \Big\{[X_0 : X_1 : X_2] \in \mathbb{CP}^2\colon X_0^2 + X_1^2 + X_2^2 = 0\Big\},$$
$$Q_2 = \Big\{[X_0 : X_1 : X_2] \in \mathbb{CP}^2\colon X_0X_1 + X_1X_2 + X_2X_0 = 0\Big\}.$$

Is $Q_1$ transverse to $Q_2$. What can you say about $Q_1 \cap Q_2$?

### Solutions to exercises

**Exercise 9.6.** On $V_+$ and $V_-$, the vector fields $\xi_+$ and $\xi_-$ vanish only at the origin of $V_+$ and $V_-$, respectively. Under the chart maps $\varphi_\pm$, these correspond to two distinct points on $S^2$, namely the north and south poles. So, the global vector field $\xi$ has two zeros.

To determine whether $\xi$ is transverse to the zero section, we examine the matrix of partial derivatives of the coefficients of $\xi$ in the corresponding charts at each zero.

At the origin of $V_+$, the matrix of partial derivatives of the coefficients of $\xi_+$ is

$$\begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} \\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} \end{bmatrix} = \mathrm{I}_2.$$

Similarly, at the origin of $V_-$, the matrix of partial derivatives of the coefficients of $\xi_-$ is $-\mathrm{I}_2$.

Therefore, $\xi$ is a transverse section of the tangent bundle. . $\square$

**Exercise 9.7.** Recall that $\mathbb{CP}^1$ can be covered by two copies of $\mathbb{C}$, $V_0 = \mathbb{C}$ and $V_1 = \mathbb{C}$, with the following gluing data:

- $V_{0,1}, V_{1,0} = \mathbb{C}^*$;
- the transition map $\varphi_{0\mapsto 1} \colon \mathbb{C}^* \to \mathbb{C}^*$ is given by $z \mapsto w = z^{-1}$, where $z$ is the coordinate on $V_0$ and $w$ is the coordinate on $V_1$.

As we showed in the solution to Exercise 6.12, the local holomorphic vector field $\xi_0 = (a + bz + cz^2)\partial_z$ on $V_0$ matches the local holomorphic vector field $\xi_1 = -(aw^2 + bw + c)\partial_w$ on $V_1$.

If $c \neq 0$, then $w = 0$ is not a zero of $\xi_1$, and therefore the only zeros are on $V_0$. Furthermore, the quadratic polynomial $a + bz + cz^2$ has either two distinct roots or a double root, depending on whether $b^2 - 4ac \neq 0$ or not. If it has a double root $z_0$, then both the polynomial and its derivative vanish at $z_0$, so $\xi_0$ (and hence $\xi$) is not transverse to the zero section.

Symmetrically, if $a \neq 0$, then $z = 0$ is not a zero of $\xi_0$, and therefore the only zeros are on $V_1$. The polynomial $aw^2 + bw + c$ again has either two distinct roots or a double root depending on whether $b^2 - 4ac \neq 0$. If it has a double root $w_0$, then both the polynomial and its derivative vanish at $w_0$, so $\xi_1$ (and hence $\xi$) is not transverse to the zero section.

Finally, if $a = c = 0$ and $b \neq 0$, then $\xi_0$ and $\xi_1$ vanish at $z = 0$ and $w = 0$, respectively, and both zeros are transverse.

We conclude that $\xi$ is a transverse section if and only if $b^2 - 4ac \neq 0$. In this case, $\xi$ has two distinct zeros; otherwise, it has a non-transverse double zero. □

**Exercise 9.8.** Recall that

$$T_{(p,p)}\Delta_M = \{(v,v) \in T_{(p,p)}(M \times M) \colon v \in T_pM\}.$$

Also,

$$\begin{aligned} T_{(p,f(p))}M_f &= \operatorname{image}\Big(d_{(p,p)}(\mathrm{id} \times f)\Big) \\ &= \big\{(v, d_pf(v)) \in T_{(p,f(p))}(M \times M) \colon v \in T_pM\big\}. \end{aligned}$$

Clearly, $M_f \cap \Delta_M$ corresponds to the subset $\operatorname{Fix}(f) \subset M$ of fixed points of $f$.

If $M_f$ and $\Delta_M$ are transverse, then by Definition 9.1,

$$T_{(p,p)}M_f + T_{(p,p)}\Delta_M = T_{(p,p)}(M \times M),$$

for every $p \in \operatorname{Fix}(f)$. Since both summands on the left have dimension $m = \dim M$, and the right-hand side has dimension $2m$, the equation above implies

$$T_{(p,p)}M_f \cap T_{(p,p)}\Delta_M = 0.$$

In other words, for $0 \neq v \in T_pM$, the vector $(v, d_pf(v)) \in T_{(p,p)}M_f$ does not belong to $T_{(p,p)}\Delta_M$; that is, $v \neq d_pf(v)$ for all $v \neq 0$.

We conclude that the differential $d_pf \colon T_pM \to T_pM$ has no eigenvalue equal to 1. The argument is reversible, giving the converse direction as well. □

**Exercise 9.9.** Since $S^3$ is defined by

$$g(z_1, z_2) = |z_1|^2 + |z_2|^2 = z_1\overline{z}_1 + z_2\overline{z}_2 = 1,$$

the tangent space $T_{(z_1,z_2)}S^3$ is the kernel of the real linear map

$$dg = z_1\, d\overline{z}_1 + \overline{z}_1\, dz_1 + z_2\, d\overline{z}_2 + \overline{z}_2\, dz_2.$$

The derivative of $f$,

$$d_zf \colon T_z\mathbb{C} \to T_{(z^a,z^b)}\mathbb{C}^2,$$

maps $\partial_z$ to $az^{a-1}\partial_{z_1} + bz^{b-1}\partial_{z_2}$.

By Definition 9.1, we want to show the equality of real vector spaces

$$\operatorname{image}(d_zf) + T_{(z^a,z^b)}S^3 = T_{(z^a,z^b)}\mathbb{C}^2, \tag{9.1}$$

whenever $|z|^{2a} + |z|^{2b} = 1$ (in particular, $z \neq 0$ on $\operatorname{image}(f) \cap S^3$).

For $z \neq 0$, we compute

$$dg\left(az^{a-1}\partial_{z_1} + bz^{b-1}\partial_{z_2}\right) = a\overline{z}^{a-1}z^a + b\overline{z}^{b-1}z^b = \overline{z}^{-1}(a|z|^{2a} + b|z|^{2b}) \neq 0.$$

Therefore, for $z \in f^{-1}(S^3)$, we have

$$az^{a-1}\partial_{z_1} + bz^{b-1}\partial_{z_2} \notin T_{(z^a,z^b)}S^3,$$

implying (9.1).

The equation $|z|^{2a} + |z|^{2b} = 1$ defines the circle $S^1(r)$ in $\mathbb{C}$ of radius $r$ such that $r^{2a} + r^{2b} = 1$. By transversality and since $f$ is an embedding away from 0 (to prove that it is one-to-one we need $\gcd(a, b) = 1$), the intersection of the image of $f$ with $S^3$ is a 1-dimensional submanifold $L$ of $S^3$. The embedding $f|_{\mathbb{C}\setminus\{0\}}$ maps $S^1(r)$ onto $L$. Therefore, $L$ is a knot in $S^3$ that depends on $a$ and $b$. $\square$

**Exercise 9.10.** Recall that $X_0, X_1, X_2$ are not actual coordinates on $\mathbb{CP}^2$, and the equations defining $Q_1$ and $Q_2$ are not functions on $\mathbb{CP}^2$. To describe them as level sets, we must restrict to charts where the equations become actual functions.

Recall from the solution to Exercise 8.16 that $\mathbb{CP}^2$ can be covered by three charts

$$\varphi_i \colon U_i \to V_i \cong \mathbb{C}^2, \quad \varphi_i\big([X_0 : X_1 : X_2]\big) = \left(x_j = \frac{X_j}{X_i}\right)_{j\neq i}, \quad \text{for } i = 0, 1, 2.$$

To visually distinguish the coordinates on $V_0$, $V_1$, and $V_2$, let

- $x_1 = \frac{X_1}{X_0}$ and $x_2 = \frac{X_2}{X_0}$ denote the coordinates on $V_0$;
- $y_0 = \frac{X_0}{X_1}$ and $y_2 = \frac{X_2}{X_1}$ denote the coordinates on $V_1$;
- $z_0 = \frac{X_0}{X_2}$ and $z_1 = \frac{X_1}{X_2}$ denote the coordinates on $V_2$;

The coordinates $(x_1, x_2)$ and $(y_0, y_2)$ are related by

$$x_1 = \frac{1}{y_0}, \quad x_2 = \frac{y_2}{y_0}.$$

Similarly, the coordinates $(x_1, x_2)$ and $(z_0, z_1)$ are related by

$$x_1 = \frac{z_1}{z_0}, \quad x_2 = \frac{1}{z_0}.$$

The equations of $\varphi_0(Q_1 \cap U_0)$, $\varphi_1(Q_1 \cap U_1)$, and $\varphi_2(Q_1 \cap U_2)$ in $V_0$, $V_1$, and $V_2$, respectively, are

$$\begin{aligned} 1 + x_1^2 + x_2^2 &= 0, \\ y_0^2 + 1 + y_2^2 &= 0, \\ z_0^2 + z_1^2 + 1 &= 0. \end{aligned}$$

So they all represent the same equation $f(x, y) = 1 + x^2 + y^2 = 0$ in $\mathbb{C}^2$. We compute

$$df = 2x\, dx + 2y\, dy.$$

This derivative vanishes only at $(x, y) = (0, 0)$, which does not satisfy $f(x, y) = 0$. Therefore, 0 is a regular value of $f$, and the level set $\{1 + x^2 + y^2 = 0\}$ defines a holomorphic submanifold of $\mathbb{C}^2$. We conclude that $Q_1$ is a one-dimensional holomorphic submanifold of $\mathbb{CP}^2$.

Similarly, the equations of $\varphi_0(Q_2 \cap U_0)$, $\varphi_1(Q_2 \cap U_1)$, and $\varphi_2(Q_2 \cap U_2)$ in $V_0$, $V_1$, and $V_2$, respectively, are

$$\begin{aligned}
&x_1 + x_1x_2 + x_2 = 0,\\
&y_0 + y_2 + y_0y_2 = 0,\\
&z_0z_1 + z_1 + z_0 = 0,
\end{aligned}$$

so they all represent the same equation $g(x, y) = x + xy + y = 0$ in $\mathbb{C}^2$. We compute

$$dg = (1 + y)\,dx + (1 + x)\,dy.$$

This derivative vanishes only at $(x, y) = (-1, -1)$, which does not satisfy $g(x, y) = 0$. Hence, 0 is a regular value of $g$, and the level set $\{x+xy+y = 0\}$ defines a holomorphic submanifold of $\mathbb{C}^2$. We conclude that $Q_2$ is a one-dimensional holomorphic submanifold of $\mathbb{CP}^2$.

To determine whether $Q_1$ and $Q_2$ intersect transversely, it suffices to check their intersection in one chart at a time. Consider their intersection in $V_0$:

$$(x + xy + y = 0) \quad \text{and} \quad (1 + x^2 + y^2 = 0)$$

in $\mathbb{C}^2$. First, solve

$$x + xy + y = 0 \quad \Leftrightarrow \quad (x + 1)(y + 1) = 1.$$

Setting $u = x + 1$ and thus $y + 1 = 1/u$, the second equation becomes

$$1 + (u - 1)^2 + \left(\frac{1}{u} - 1\right)^2 = 0 \quad \Leftrightarrow \quad u^4 - 2u^3 + 3u^2 - 2u + 1 = 0.$$

The roots of this quartic are

$$\mu = \frac{1 \pm \mathrm{i}\sqrt{3}}{2},$$

each with multiplicity 2. The presence of multiplicities indicates that $Q_1$ and $Q_2$ do not intersect transversely. The corresponding points in $(x, y)$-coordinates are

$$(x, y) = p_\mu := \big(\mu - 1, \overline{\mu} - 1\big).$$

At each point $p_\mu$, we compute the differentials:

$$\begin{aligned}
d(x + xy + y)\big|_{p_\mu} &= (1 + y)\,dx + (1 + x)\,dy\big|_{p_\mu} = \overline{\mu}\,dx + \mu\,dy,\\
d(1 + x^2 + y^2)\big|_{p_\mu} &= 2x\,dx + 2y\,dy\big|_{p_\mu} = (2\mu - 2)\,dx + (2\overline{\mu} - 2)\,dy.
\end{aligned}$$

These two differentials are proportional because $\overline{\mu} = \frac{-1}{2}(2\mu - 2)$. Therefore,

$$T_{p_\mu}Q_1 + T_{p_\mu}Q_2 = \ker\big(\overline{\mu}\,dx + \mu\,dy\big) + \ker\big((2\mu - 2)\,dx + (2\overline{\mu} - 2)\,dy\big) \cong \mathbb{C} \neq T_{p_\mu}\mathbb{C}^2.$$

The calculations in other two charts are identical. In conclusion, as the calculations indicate, $Q_1$ and $Q_2$ non-transversely intersect at two points (with multiplicity two), each lying in the domain of all three charts.

$\square$

Chapter 10

# Constant Rank and Whitney Embedding Theorems

In this lecture, we'll discuss the proof of the Constant Rank Theorem and then state and prove Whitney's Embedding Theorem regarding the embedding of smooth manifolds into Euclidean spaces.

Constant Rank Theorem is a local statement, and manifolds are locally Euclidean. Therefore, we need to prove the following local result.

**Theorem 10.1.** *Suppose $f\colon U \to U'$ is a smooth (respectively, holomorphic) map of constant rank $r$ between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^{m'}$ (respectively, $\mathbb{C}^m$ and $\mathbb{C}^{m'}$). Then, for every $p \in U$, after possibly shrinking $U$ and $U'$ around $p$ and $f(p)$, there exist coordinate systems*[1] *$(x_1, \ldots, x_m)$ on $U$ and $(y_1, \ldots, y_{m'})$ on $U'$ such that*

$$f(x_1, \ldots, x_m) = (x_1, \ldots, x_r, \underbrace{0, \ldots, 0}_{\text{if } r<m'}).$$

The proof uses the Inverse Function Theorem stated below, which we will not prove here.

[1] This is the pedestrian word for charts.

**Theorem 10.2** (Inverse Function Theorem)**.** *Suppose $f\colon U \to U'$ is a smooth (respectively, holomorphic) map between open subsets of $\mathbb{R}^m$ (respectively, $\mathbb{C}^m$). If the linear map*

$$d_p f\colon T_pU \to T_{f(p)}U'$$

*is invertible, then $f$ is smoothly (respectively, holomorphically) invertible near $p$. In other words, after possibly shrinking $U$ and $U'$ around $p$ and $f(p)$, there exists $g\colon U' \to U$ such that $g$ is smooth (respectively, holomorphic) and $g \circ f = \mathrm{id}$.*

**Exercise 10.3.** Deduce the holomorphic version of Inverse Function Theorem from the smooth version.

**Proof of Theorem 10.1.** Starting with the standard coordinates on $\mathbb{R}^m$ and $\mathbb{R}^{m'}$ (respectively, $\mathbb{C}^m$ and $\mathbb{C}^{m'}$), and after possibly shrinking $U$ and $U'$, the goal is to construct diffeomorphisms (respectively, biholomorphisms)

$$\varphi\colon U \to V \subset \mathbb{R}^m, \quad \varphi'\colon U' \to V' \subset \mathbb{R}^{m'}$$

(respectively, $\varphi\colon U \to V \subset \mathbb{C}^m$, $\varphi'\colon U' \to V' \subset \mathbb{C}^{m'}$) such that

$$\varphi' \circ f \circ \varphi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, \underbrace{0, \ldots, 0}_{\text{if } r<m'}).$$

Without loss of generality, we may assume $p = 0 \in U$ and $f(p) = 0 \in U'$. Also, after a linear change of coordinates, we may assume

$$d_0 f = \begin{bmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{bmatrix}.$$

If

$$f(x) = (f_1(x), \ldots, f_{m'}(x)),$$

define

$$\varphi\colon U \to \mathbb{R}^m, \qquad \varphi(x_1, \ldots, x_m) = (f_1(x), \ldots, f_r(x), x_{r+1}, \ldots, x_m).$$

Then the derivative at the origin is

$$d_0\varphi = \begin{bmatrix} \mathrm{I}_r & \star \\ 0 & \mathrm{I}_{m-r} \end{bmatrix}.$$

Therefore, by the Inverse Function Theorem, after possibly shrinking $U$ around 0, the map $\varphi$ is a diffeomorphism (respectively, biholomorphism) from $U$ onto some open set $V \subset \mathbb{R}^m$ (respectively, $V \subset \mathbb{C}^m$). By construction,

$$f \circ \varphi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, h_{r+1}(x), \ldots, h_{m'}(x)),$$

for some smooth (respectively, holomorphic) functions $h_{r+1}, \ldots, h_{m'}$.

Since

$$\operatorname{rank}\ d(f \circ \varphi^{-1}) = \operatorname{rank}\ df = r,$$

and

$$d(f \circ \varphi^{-1}) = \begin{bmatrix} \mathrm{I}_r & 0 \\ \star & \left[\frac{\partial h_i}{\partial x_j}\right]_{r<i\le m',\ r<j\le m} \end{bmatrix},$$

we conclude that

$$\left[\frac{\partial h_i}{\partial x_j}\right]_{r<i\le m',\ r<j\le m} \equiv 0.$$

In other words, for fixed $(x_1, \ldots, x_r)$, the function $h_i(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m)$ is constant in $(x_{r+1}, \ldots, x_m)$.

After possibly shrinking $U'$ around the origin, define

$$\varphi' \colon U' \to \mathbb{R}^{m'},$$
$$(y_1, \ldots, y_{m'}) \mapsto \Big(y_1, \ldots, y_r, (y_i - h_i(y_1, \ldots, y_r, 0, \ldots, 0))_{r<i\le m'}\Big).$$

Then the derivative at the origin is

$$d_0\varphi' = \begin{bmatrix} \mathrm{I}_r & 0 \\ \star & \mathrm{I}_{m'-r} \end{bmatrix}.$$

Hence, by the Inverse Function Theorem, after possibly shrinking $U'$, the map $\varphi'$ is a diffeomorphism (respectively, biholomorphism) from $U'$ onto some open set $V' \subset \mathbb{R}^{m'}$ (respectively, $V' \subset \mathbb{C}^{m'}$).

It is now easy to check that

$$\varphi' \circ f \circ \varphi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0).$$

□

In previous lectures, we learned a method for constructing interesting manifolds inside simple ambient spaces such as $\mathbb{R}^n$ and $\mathbb{C}^n$ by considering level sets of non-trivial functions defined on these spaces. The following theorem shows that, in fact, any smooth manifold can be embedded into some $\mathbb{R}^n$ for sufficiently large $n$. The holomorphic analogue of this statement is certainly not true, since the only holomorphic functions on closed holomorphic manifolds are the constant functions. However, some holomorphic manifolds can be embedded into $\mathbb{CP}^n$ or open subsets of that, and these are called algebraic varieties. Algebraic geometry uses algebraic methods to study such holomorphic manifolds and generalizes them extensively to include a wide range of singular spaces and more abstractly defined geometric objects.

**Theorem 10.4.** *Any smooth $m$-manifold $M$ can be embedded into $\mathbb{R}^{2m}$. Moreover, if $m \ge 2$, it can be immersed into $\mathbb{R}^{2m-1}$.*

We will not prove this full version in this lecture and instead refer the reader to [**Hir76**, Theorems 8.4.1, 8.4.2], as we do not need it here and the proof is quite long and technical. Instead, we present the following simplified version, whose proof is one of several applications of partition of unity. The proof of

Theorem 10.4 will rely on this simpler version, along with an exhaustion of the manifold by compact subsets and some dimension reduction arguments.

**Theorem 10.5** (Simplified Version of the Whitney Embedding Theorem). *Every compact smooth manifold $M$ admits an embedding into $\mathbb{R}^k$ for some sufficiently large $k$.*

**Proof.** Since the manifold is compact, we need to build a one-to-one immersion. Around every point $p \in M$, there is a chart $\varphi_p \colon U_p \longrightarrow V_p \subset \mathbb{R}^m$ (or $\mathbb{H}_m$) such that $\varphi_p(p) = 0$, $V_p = B_3(0)$ (or $B_3(0) \cap \mathbb{H}_m$). Let $\varrho_p \colon U_p \longrightarrow [0,1]$ be a smooth function such that

$$\varrho_p\big|_{\varphi_p^{-1}(B_1(0))} \equiv 1 \quad \text{and} \quad \varrho_p\big|_{U_p - \varphi_p^{-1}(B_2(0))} \equiv 0.$$

By the second property, each $\varphi_p$ trivially extends to a function on the entire $M$. Note that

$$\varrho_p \cdot \varphi_p \colon M \longrightarrow \mathbb{R}^m$$

is well-defined, smooth, and identically equal to $0 \in \mathbb{R}^m$ outside $\varphi_p^{-1}(B_2(0))$.

Since $M$ is compact, we can choose finitely many of these charts, say indexed by $\{p_1, \ldots, p_\ell\}$, such that $\left\{\varphi_{p_i}^{-1}(B_1(0))\right\}_{i=1}^{\ell}$ is an open cover of $M$.

Define

$$f = \left(\varrho_{p_1}, \ldots, \varrho_{p_\ell}, \varphi_{p_1} \cdot \varphi_{p_1}, \ldots, \varrho_{p_\ell} \cdot \varphi_{p_\ell}\right) \colon M \longrightarrow \mathbb{R}^{\ell(m+1)}.$$

Suppose $f(q_1) = f(q_2)$. By assumption, there exists $i = 1, \ldots, \ell$ such that $q_1 \in \varphi_{p_i}^{-1}(B_1(0))$. Therefore, $\varrho_{p_i}(q_2) = \varrho_{p_i}(q_1) = 1$. We conclude that $q_1, q_2 \in \varphi_{p_i}^{-1}(B_2(0))$ and

$$\varphi_{p_i}(q_1) = \varphi_{p_i}(q_2).$$

Therefore, $q_1 = q_2$. We have shown that $f$ is one-to-one.

Moreover, for every $q \in M$, there exists $i = 1, \ldots, \ell$ such that $q \in \varphi_{p_i}^{-1}(B_1(0))$. Therefore,

$$\varrho_{p_i} \cdot \varphi_{p_i}(q) = \varphi_{p_i}(q).$$

We conclude that $f$ is an immersion at $q$ (because $f \circ \varphi_{p_i}^{-1} = \mathrm{id}_{\mathbb{R}^m}$ on $B_1(0)$). $\square$

## Solutions to exercises

**Exercise 10.3.** Without loss of generality, we may assume $p = f(p) = 0$. Thus, suppose

$$f\colon U \to U', \qquad f\big(z = (z_1, \ldots, z_m)\big) = (f_1, \ldots, f_m)$$

is a holomorphic map between open neighborhoods of 0 in $\mathbb{C}^m$, and the complex linear map

$$d_0 f\colon T_0 U \to T_0 U', \qquad d_0 f = \left[\frac{\partial f_i}{\partial z_j}(0)\right]_{1 \le i,j \le m}$$

is invertible. We can think of $f$ as a smooth map between open sets of $\mathbb{R}^{2m}$ by decomposing each $z_j$ and $f_i$ into their real and imaginary parts:

$$f(x,y) = \big(g_1(z), \ldots, g_m(z), h_1(z), \ldots, h_m(z)\big),$$

where

$$z = x + \mathrm{i}y, \quad f(z) = g(z) + \mathrm{i}h(z).$$

Then the real Jacobian of $f$ with respect to the $x, y$-variables is

$$d_0^{\mathbb{R}} f = \begin{bmatrix} \left[\frac{\partial g_i}{\partial x_j}(0)\right]_{1 \le i,j \le m} & \left[\frac{\partial g_i}{\partial y_j}(0)\right]_{1 \le i,j \le m} \\ \left[\frac{\partial h_i}{\partial x_j}(0)\right]_{1 \le i,j \le m} & \left[\frac{\partial h_i}{\partial y_j}(0)\right]_{1 \le i,j \le m} \end{bmatrix}.$$

The determinants of $d_0^{\mathbb{R}} f$ and $d_0 f$ are related by

$$\det(d_0^{\mathbb{R}} f) = |\det(d_0 f)|^2.$$

Therefore, by the smooth version of the Inverse Function Theorem, $f$ is locally smoothly invertible near the origin.

Since

$$f^{-1}(f(z)) = z,$$

differentiating both sides with respect to $\overline{z}_i$ gives

$$0 = \frac{\partial}{\partial \overline{z}_i} f^{-1}(f(z)) = \sum_{j=1}^{m} \frac{\partial f^{-1}}{\partial z_j} \frac{\partial f_j}{\partial \overline{z}_i} + \sum_{j=1}^{m} \frac{\partial f^{-1}}{\partial \overline{z}_j} \frac{\partial \overline{f_j}}{\partial \overline{z}_i}.$$

Since $\frac{\partial f_j}{\partial \overline{z}_i} = 0$, we obtain

$$\sum_{j=1}^{m} \frac{\partial f^{-1}}{\partial \overline{z}_j} \overline{\frac{\partial f_j}{\partial z_i}} = 0.$$

Moreover, since the matrix $\left[\frac{\partial f_j}{\partial z_i}(0)\right]_{1 \le i,j \le m}$ is invertible, we conclude

$$\frac{\partial f^{-1}}{\partial \overline{z}_j} = 0 \qquad \forall\ 1 \le j \le m,$$

i.e., $f^{-1}$ is holomorphic as well.

Chapter 11

# Vector bundles

We have already encountered an example of a vector bundle – namely, the tangent bundle. More generally, a vector bundle is a family of vector spaces parametrized by the points of a manifold, such that the family is locally trivial. The precise definition is as follows.

**Definition 11.1.** A continuous, smooth, or holomorphic vector bundle consists of

- A pair of continuous, smooth, or holomorphic manifolds $E$ and $M$, and a surjective map $\pi\colon E \longrightarrow M$ of the same class,
- A real or complex vector space structure on each fiber $E_p = \pi^{-1}(p)$ for all $p \in M$,

such that, for each $p \in M$, there exists an open neighborhood $U \ni p$ and a $C^0$, smooth, or holomorphic identification (called **local trivialization**)

$$\Phi\colon E|_U = \pi^{-1}(U) \longrightarrow U \times F \quad (F \cong \mathbb{R}^k \text{ or } \mathbb{C}^k),$$

that maps each fiber $E_p$ linearly isomorphically onto $\{p\} \times F$.

**Example 11.2.** The simplest example of a vector bundle is a **trivial (i.e., product) bundle** $E = M \times F$, for some fixed vector space $F$, where $\pi$ is the projection map onto the first factor. By definition, every vector bundle is locally trivial.

**Remark 11.3.** (1) There are slightly different ways to define a vector bundle in the literature, but they all describe the same class of objects.

(2) Every holomorphic vector bundle has complex vector spaces as its fibers, but there are smooth or continuous vector bundles with complex vector spaces as their fibers that are not holomorphic because $M$ or $\pi$ is not holomorphic – for instance, $S^1 \times \mathbb{C} \longrightarrow S^1$.

(3) In practice, we often assume that the open sets are domains of chart maps $\varphi\colon U \longrightarrow V \subset A$. Since $\varphi$ identifies $U$ with $V$, one may instead define $\Phi$ to be an identification between $E|_U$ and $V \times F$ lifting the map $\varphi$; that is, a commutative diagram

$$\begin{tikzcd} E|_U \arrow[r, "\Phi"] \arrow[d, "\pi"] & V \times F \arrow[d] \\ U \arrow[r, "\varphi"] & V \end{tikzcd}.$$

More precisely, the composition of $\Phi\colon E|_U \longrightarrow U \times F$ and $\varphi \times \mathrm{id}_F\colon U \times F \longrightarrow V \times F$ gives the identification

$$(\varphi \times \mathrm{id}_F) \circ \Phi\colon E|_U \longrightarrow V \times F$$

fitting the commutative diagram above. Moreover, since $\pi$ is continuous, $E|_U = \pi^{-1}(U)$ is an open subset of $E$, and every local trivialization $E|_U \longrightarrow V \times F \subset A \times F$ is indeed a chart on $E$.

(4) Similarly to Remark 1.5.2, in some examples, we may reverse the arrows and define a local trivialization to be an identification

$$\Phi\colon U \times F \to E|_U \quad \text{or} \quad \begin{tikzcd} V \times F \arrow[r, "\Phi"] \arrow[d, "\pi"] & E|_U \arrow[d] \\ V \arrow[r, "\varphi"] & U \end{tikzcd}.$$

We will switch between these different perspectives whenever it helps simplify the notation.

(5) The integer $\dim F$ (over $\mathbb{R}$ or $\mathbb{C}$) is the same for all local trivializations (on a connected manifold) and is called the **rank** of vector bundle.

Given a vector bundle $E \longrightarrow M$, suppose $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ is an open covering of $M$ and

$$\Phi_\alpha\colon E|_{U_\alpha} \longrightarrow U_\alpha \times F_\alpha \qquad \forall\, \alpha \in \mathcal{I}$$

is a collection of local trivializations. Then the transition maps

$$\Phi_\beta \circ \Phi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times F_\alpha \longrightarrow (U_\alpha \cap U_\beta) \times F_\beta$$

are of the form $(x, v) \mapsto (x, \Phi_{\alpha \mapsto \beta}(x)(v))$ for some $x$-dependent family of linear isomorphisms

$$\Phi_{\alpha \mapsto \beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{Isom}(F_\alpha, F_\beta).$$

Fixing an identification $F_\alpha = \mathbb{R}^k$ or $\mathbb{C}^k$ for all $\alpha \in \mathcal{I}$, the latter is simply a matrix-valued function

$$\Phi_{\alpha\mapsto\beta}\colon U_{\alpha,\beta} \longrightarrow \mathrm{GL}(k,\mathbb{R}) \text{ or } \mathrm{GL}(k,\mathbb{C}). \tag{11.1}$$

**Remark 11.4.** If we change to the perspective of the third item in Remark 11.3, then

$$\Phi_\alpha\colon E|_{U_\alpha} \longrightarrow V_\alpha \times F_\alpha \qquad \forall\ \alpha \in \mathcal{I}$$

and the transition maps

$$\Phi_\beta \circ \Phi_\alpha^{-1}\colon V_{\alpha,\beta} \times F_\alpha \longrightarrow V_{\beta,\alpha} \times F_\beta$$

are lifts of

$$\varphi_{\alpha\mapsto\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}\colon V_{\alpha,\beta} \longrightarrow V_{\beta,\alpha}$$

mapping $(x,v)$ to $(\varphi_{\alpha\mapsto\beta}(x), \Phi_{\alpha\mapsto\beta}(x)(v))$ such that

$$\Phi_{\alpha\mapsto\beta}\colon V_{\alpha,\beta} \longrightarrow \mathrm{Isom}(F_\alpha, F_\beta)$$

is a family of linear isomorphisms. We will switch between these two perspectives whenever it helps simplify the notation. In the following, we use the second point of view.

In Section 4, we learned that we can ignore the chart maps and focus on the transition functions to construct $M$ as a quotient space obtained by gluing affine pieces via transition maps. The same can be done for vector bundles as follows. This point of view will sometimes make it easier to justify why certain examples are vector bundles. It also reveals the information needed to characterize a vector bundle.

If a manifold is described as a quotient space

$$M = \coprod_{\alpha\in\mathcal{I}} V_\alpha/\sim, \qquad x \sim y \quad \Leftrightarrow \quad x \in V_{\alpha,\beta},\ y \in V_{\beta,\alpha},\ y = \varphi_{\alpha\mapsto\beta}(x),$$

a vector bundle $E$ on $M$ that is locally trivial on the image of each $V_\alpha$ is a manifold of a similar qoutient form

(11.2)

$$E = \coprod_{\alpha\in\mathcal{I}} (V_\alpha \times F_\alpha)/\sim, \quad \text{such that } (x,v) \sim (y,w) \Leftrightarrow$$

$$(x,v) \in V_{\alpha,\beta} \times F_\alpha,\ (y,w) \in V_{\beta,\alpha} \times F_\beta,\ (y,w) = \big(\varphi_{\alpha\mapsto\beta}(x), \Phi_{\alpha\mapsto\beta}(x)(v)\big),$$

where, for each $x$,

$$\Phi_{\alpha\mapsto\beta}(x) \in \mathrm{Isom}(F_\alpha, F_\beta)$$

is a linear isomorphism between $F_\alpha$ and $F_\beta$. In other words, $\Phi_{\alpha\mapsto\beta}$ is a continuous, smooth, or holomorphic map

$$\Phi_{\alpha\mapsto\beta}\colon V_{\alpha,\beta} \longrightarrow \mathrm{Isom}(F_\alpha, F_\beta).$$

Fixing an identification $F_\alpha = \mathbb{R}^k$ or $\mathbb{C}^k$ for all $\alpha \in \mathcal{I}$, the latter is simply a matrix-valued function

$$\Phi_{\alpha\mapsto\beta} \colon V_{\alpha,\beta} \longrightarrow \mathrm{GL}(k,\mathbb{R}) \text{ or } \mathrm{GL}(k,\mathbb{C}).$$

These identifications are required to satisfy $\Phi_{\alpha\alpha} = \mathrm{id}_{F_\alpha}$, for all $\alpha \in \mathcal{I}$, and the cocyle condition illustrated by the commutative diagram

$$\begin{tikzcd}
V_{\alpha,\beta\gamma} \times F_\alpha \arrow[r, "\Phi_{\alpha\mapsto\beta}"] \arrow[d] \arrow[rr, bend left, "\Phi_{\alpha\mapsto\gamma}"] & V_\beta \times F_\beta \arrow[r, "\Phi_{\beta\mapsto\gamma}"] \arrow[d] & V_\gamma \times F_\gamma \arrow[d] \\
V_{\alpha,\beta\gamma} \arrow[r, "\varphi_{\alpha\mapsto\beta}"] \arrow[rr, bend right, "\varphi_{\beta\mapsto\gamma}"'] & V_\beta \arrow[r, "\varphi_{\beta\mapsto\gamma}"] & V_\gamma\,.
\end{tikzcd}$$

In other words,

$$\Phi_{\alpha\mapsto\gamma}(x) = \Phi_{\alpha\mapsto\gamma}(\varphi_{\alpha\mapsto\beta}(x)) \circ \Phi_{\alpha\mapsto\gamma}(x) \quad \forall\, x \in V_{\alpha,\beta\gamma}. \tag{11.3}$$

By the definition of $\sim$, the projection maps $\pi_\alpha \colon V_\alpha \times F_\alpha \longrightarrow V_\alpha$ are compatible with respect to $\sim$ and patch together to define the projection map $\pi \colon E \longrightarrow M$. Also, since $M$ is Hausdorff by assumption, the quotient space (11.2) is automatically Hausdorff.

Just as in the example of tangent space, every $x \in V_\alpha$, the fiber $\pi_\alpha^{-1}(x) = \{x\} \times F_\alpha$ is a vector space identified with $F_\alpha$, and if $x \in V_\alpha$ is equivalent to $y \in V_\beta$, then

$$\Phi_{\alpha\mapsto\beta}(x) \colon \{x\} \times F_\alpha \longrightarrow \{y\} \times F_\beta$$

is a linear isomorphism. Therefore, each fiber of $E$ has a well-defined vector space structure. However, the particular identification of that with $\mathbb{R}^m$ or $\mathbb{C}^m$ depends on the choice of a basis.

The following analogue of Lemma 4.1 holds for similar reasons.

**Lemma 11.5.** *There is a one-to-one correspondence between vector bundles presented as* (11.2) *and the pairs consisting of a vector bundle $E \longrightarrow M$ and a collection of local trivializations over a countable atlas of $M$.*

**Example 11.6.** For every smooth or holomorphic manifold $M$, the tangent bundle $TM$ is a vector bundle of rank equal to the dimension of $M$. In this example, and from the point of view of (11.2), the linear isomorphisms $\Phi_{\alpha\mapsto\beta}$ are simply the derivatives $d\varphi_{\alpha\mapsto\beta}$ of the transition functions $\varphi_{\alpha\mapsto\beta}$; see Definition 6.3. The cocycle condition corresponds to the chain rule.

**Example 11.7.** A **line bundle** is a real or complex vector bundle of rank one. In this case, if we have $F_\alpha = \mathbb{R}$ or $\mathbb{C}$ for all $\alpha \in \mathcal{I}$, then $\Phi_{\alpha\mapsto\beta}$ is simply a nowhere-vanishing function

$$\Phi_{\alpha\mapsto\beta}\colon V_{\alpha,\beta} \longrightarrow \mathbb{R}^* = \mathrm{Isom}(\mathbb{R},\mathbb{R}) \text{ or } \mathbb{C}^* = \mathrm{Isom}(\mathbb{C},\mathbb{C}).$$

This makes working with line bundles much easier than with arbitrary vector bundles, where matrix multiplications are non-commutative.

**Definition 11.8.** Given a vector bundle $\pi\colon E \longrightarrow M$, a **section** of $E$ is a map $s\colon M \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_M$.

For instance, a section of tangent bundle is a vector field on the manifold. Sections generalize the concept of graph of functions in Calculus and naturally arise in many contexts such as in the example of vector fields on smooth manifolds. As another example of their importance, recall that closed holomorphic manifolds do not admit any non-constant holomorphic function. However, many of them admit complex line bundles (i.e. rank 1 vector bundle) with plenty of sections. These sections can be used to embed the manifold into a projective space or to define interesting holomorphic submanifolds. For instance the equations in Exercise 9.10 can be realized as sections of a holomorphic line bundle on $\mathbb{CP}^2$.

**Definition 11.9.** Given a rank $r$ vector bundle $\pi\colon E \longrightarrow M$ and $Y \subset M$, a frame for $E|_Y$ is a set of $r$ sections $s_1,\dots,s_r\colon Y \longrightarrow E|_Y$ such that $\{s_1(p),\dots,s_r(p)\}$ is a basis for $E_p$ at every $p \in Y$.

**Lemma 11.10.** *Let $\pi\colon E \to M$ be a real or complex vector bundle, and let $U \subset M$ be an open set. Then there is a one-to-one correspondence between local trivializations*

$$\Phi\colon E|_U \to U \times \mathbb{R}^r \quad \text{or} \quad U \times \mathbb{C}^r \tag{11.4}$$

*and real or complex frames for $E|_U$.*

**Proof.** Given a local trivialization

$$\Phi\colon E|_U \to U \times \mathbb{R}^r \quad \text{or} \quad U \times \mathbb{C}^r,$$

define sections $s_i(x) = \Phi^{-1}(x, e_i)$, where $e_i$ is the $i$-th standard basis vector. Then $\{s_1,\dots,s_r\}$ forms a frame for $E|_U$.

Conversely, given a frame $\{s_1,\dots,s_r\}$ over $U$, define a local trivialization

$$\Phi\colon U \times \mathbb{R}^r \quad \text{or} \quad U \times \mathbb{C}^r \to E|_U$$

by setting

$$\Phi(x,(a_1,\dots,a_r)) = \sum_{i=1}^r a_i s_i(x) \in E_x.$$

This defines a local trivialization in the sense of Remark 11.3.4. One may look at $\Phi^{-1}$ to get a local trivialization in the sense of (11.4). □

The trivialization maps we encountered above are special cases of vector bundle homomorphisms defined below.

**Definition 11.11.** Let $\pi\colon E \to M$ and $\pi'\colon E' \to M'$ be vector bundles. A **vector bundle homomorphism** is a commutative diagram

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

such that for every point $p \in M$, the induced map on the fibers

$$E_p \to E'_{f(p)}$$

is linear. Depending on context, the pair $(f, h)$ is assumed to be continuous, smooth, or holomorphic.

There are several important special cases, particularly when $f = \mathrm{id}\colon M \to M$ and $h$ satisfies one of the following:

(1) If $h$ is injective, it is called an **inclusion** or **embedding**, and the image of $E$ in $E'$ is a **sub vector bundle** of $E'$.
(2) If $h$ is surjective, then $E'$ is a **quotient** of $E$. We will study this case in more detail in the next section.
(3) If $h$ is an isomorphism, then $E$ and $E'$ are considered **isomorphic** as vector bundles.

**Example 11.12.** As an example of the third case, we say a vector bundle is **trivial** if it is isomorphic to a product bundle $M \times F$.

**Example 11.13.** Associated to every real or complex projective space $\mathbb{P}(V)$ there is a **tautological (real or complex) line bundle** $\gamma \longrightarrow \mathbb{P}(V)$ defined in the following way. Every point in $\mathbb{P}(V)$ corresponds to a line $\ell \subset V$. Thus, we define $\gamma$ as a sub vector bundle of the trivial bundle $\mathbb{P}(V) \times V$ by

$$\gamma = \{(\ell, v) \in \mathbb{P}(V) \times V : v \in \ell\} \tag{11.5}$$

where the projection map $\gamma \longrightarrow \mathbb{P}(V)$ is simply restriction to $\gamma$ of the canonical projection map $\mathbb{P}(V) \times V \longrightarrow V$. Let us find the transition maps of $\gamma$ and show that $\gamma$ is a smooth or holomorphic vector bundle depending on whether $V$ is a real or complex vector space. Identifying $V$ with $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$, recall from (2.8) that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ can be covered by $(n+1)$

standard charts:

$$\varphi_i\colon U_i \to V_i, \quad \varphi_i\big([X_0\colon \ \dots : X_n]\big) = \left(x_j = \frac{X_j}{X_i}\right)_{j\neq i}, \quad \text{for } i = 0, \dots, n,$$

where $U_i$ is the open subset defined by $X_i \neq 0$ and $V_i = \mathbb{R}^{\{0,\dots,\widehat{i},\dots,n\}} \cong \mathbb{R}^n$ or $V_i = \mathbb{C}^{\{0,\dots,\widehat{i},\dots,n\}} \cong \mathbb{C}^n$. Also, the transition functions of the manifold are given by

$$\varphi_{i\mapsto j} = \varphi_j \circ \varphi_i^{-1}\big((x_k)_{k\neq i}\big) = (y_k)_{k\neq j}, \quad \text{where} \quad y_k = \begin{cases} x_k/x_j & \text{if } k \neq i, \\ 1/x_j & \text{if } k = i. \end{cases}$$

By definition

$$\gamma|_{[X_0\colon \dots\colon X_n]} = \mathbb{R}\cdot(X_0\colon \ \dots : X_n) \subset \mathbb{R}^{n+1} \text{ or } \mathbb{C}\cdot(X_0\colon \ \dots : X_n) \subset \mathbb{C}^{n+1}.$$

Restricted to $U_i$, every point can be uniquely presented as

$$(x_0, \dots, x_{i-1}, x_i = 1, x_{i+1}, \dots, x_n),$$

resulting in a local trivialization

$$\Phi_i\colon U_i \times \mathbb{R} \text{ or } \mathbb{C} \longrightarrow \gamma|_{U_i},$$
$$\big((x_k)_{k\neq i}, t\big) \longrightarrow \big((x_k)_{k\neq i}, t(x_0, \dots x_{i-1}, x_i = 1, x_{i+1}, \dots, x_n)\big).$$

It is easy to calculate that

$$\Phi_{i\mapsto j}\big((x_k)_{k\neq i}\big) = x_j.$$

The latter is definitely a smooth or holomorphic non-zero function on $V_{i,j}$.

**Exercise 11.14.** Let $E' \to M$ be a smooth vector bundle, and let $E \to M$ be a subvector bundle of $E'$. Show that, over a sufficiently small neighborhood of any point in $M$, a local frame for $E$ can be extended to a local frame for $E'$.

**Exercise 11.15.** Show that the tangent bundle $TS^1$ is a trivial vector bundle, while the canonical line bundle $\gamma_n \to \mathbb{RP}^n$ is non-trivial.

**Exercise 11.16.** Show that the tangent bundle $TS^3$ is trivial.

**Exercise 11.17.** Show that the holomorphic tangent bundle $T\mathbb{CP}^1$ is not holomorphically isomorphic to the trivial complex line bundle $\mathbb{CP}^1 \times \mathbb{C}$.

The set of vector bundle homomorphisms between two vector bundles $E \longrightarrow M$ and $E' \longrightarrow M'$ that lift a fixed map $f\colon M \longrightarrow M'$ forms a module over $C^0(M)$, $C^\infty(M)$, or $C^{\text{hol}}(M)$, depending on the context. In other words, if $h_1$ and $h_2$ are two vector bundle homomorphisms as in the diagram below

$$\begin{tikzcd} E \arrow[r, shift left, "h_2"] \arrow[r, shift right, "h_1"'] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd},$$

and $g_1, g_2$ are two functions on $M$, then the linear combination $g_1 h_1 + g_2 h_2$ is also a vector bundle homomorphism that maps each $v \in E_p$ to

$$g_1(p) h_1|_p(v) + g_2(p) h_2|_p(v) \in E'_{f(p)}.$$

In the special case where $M = M'$ and $f = \mathrm{id}_M$, we denote the space of vector bundle homomorphisms from $E$ to $E'$ lifting the identity map by $\mathrm{Hom}(E, E')$. We will later show that $\mathrm{Hom}(E, E')$ is itself a vector bundle over $M$, whose fiber over $p \in M$ is the space of $\mathbb{R}$- or $\mathbb{C}$-linear maps $\mathrm{Hom}(E_p, E'_p)$ between the vector spaces $E_p$ and $E'_p$.

**Exercise 11.18.** What is the rank of vector bundle $\mathrm{Hom}(E, E') \longrightarrow M$? If $L \longrightarrow M$ is a line bundle, show that $\mathrm{Hom}(L, L) \longrightarrow M$ is trivial.

## Solutions to exercises

**Exercise 11.14.** For any point $p \in M$, fix arbitrary local frames $(s_1, \ldots, s_k)$ for $E$ and $(s'_1, \ldots, s'_{k'})$ for $E'$ over a sufficiently small neighborhood $U$ of $p$. Since $(s'_1, \ldots, s'_{k'})$ forms a basis for $E'$ at every point in $U$, there exist smooth functions $\{a_{ij}\}_{1 \le i \le k,\ 1 \le j \le k'}$ on $U$ such that

$$s_i = \sum_{j=1}^{k'} a_{ij} s'_j \qquad \text{for all } 1 \le i \le k.$$

Because the sections $(s_1, \ldots, s_k)$ are pointwise linearly independent, the matrix

$$\big[a_{ij}\big]_{1 \le i \le k,\ 1 \le j \le k'}$$

has full rank $k$ at every point in $U$. After possibly shrinking $U$ and reordering the $(s'_j)$, we may assume that the $k \times k$ minor $\big[a_{ij}\big]_{1 \le i,j \le k}$ is invertible throughout $U$. We conclude that

$$(s_1, \ldots, s_k, s'_{k+1}, \ldots, s'_{k'})$$

is a local frame for $E'|_U$. □

**Remark 11.19.** By Lemma 11.10, there is a one-to-one correspondence between local trivializations

$$\Phi\colon E|_U \to U \times \mathbb{R}^k \quad \text{and} \quad \Phi'\colon E'|_U \to U \times \mathbb{R}^{k'}$$

and local frames for $E|_U$ and $E'|_U$, respectively. Thus, the statement of the exercise is equivalent to the following: Over a sufficiently small neighborhood of any point in $M$, there exists a local trivialization of $E'$ whose restriction gives a local trivialization of $E$. We will make use this to understand the transition maps of quotient bundles.

**Exercise 11.15.** Thinking of $S^1$ as the level set of the smooth function $f(x, y) = x^2 + y^2$ on $\mathbb{R}^2$, the tangent bundle $TS^1$ is given by $\ker(df)|_{S^1}$. It is easy to see that the vector field

$$y\partial_x - x\partial_y \in \ker(df)$$

defines a nowhere-vanishing section of the real vector bundle $TS^1$. By Lemma 11.10, this section defines a trivialization of $TS^1$.

Recall that $\mathbb{RP}^n$ is the quotient space $S^n/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $S^n \subset \mathbb{R}^{n+1}$ by the antipodal map. Consider the tautological line bundle

$$\widetilde{\gamma} = \{(x, v) \in S^n \times \mathbb{R}^{n+1} : v \in \mathbb{R} \cdot x\}$$

over $S^n$. The $\mathbb{Z}_2$-action on $S^n$ lifts to a $\mathbb{Z}_2$-action on $\widetilde{\gamma}$ by

$$(x, v) \mapsto (-x, v).$$

It is clear from the definition (cf. (11.5)) that the canonical line bundle $\gamma \to \mathbb{RP}^n$ is the $\mathbb{Z}_2$-quotient of $\widetilde{\gamma}$. By Lemma 11.10, $\gamma$ is trivial if and only if it admits a nowhere-vanishing section $\xi$. Such a section would arise as the image under the quotient projection of a $\mathbb{Z}_2$-invariant, nowhere-vanishing section $\widetilde{\xi}$ of $\widetilde{\gamma}$ over $S^n$.

Consider the map

$$\zeta\colon S^n \to \widetilde{\gamma}, \quad x \mapsto (x, x),$$

which defines a nowhere-vanishing section of $\widetilde{\gamma}$. Note, however, that $(-1) \in \mathbb{Z}_2$ acts on $\zeta$ by

$$\zeta(-x) = (-x, -x) = -(x, x) = -\zeta(x),$$

so $\zeta$ is *anti*-invariant under the $\mathbb{Z}_2$-action.

Suppose we try to define a $\mathbb{Z}_2$-invariant section $\widetilde{\xi}$ by rescaling $\zeta$:

$$\widetilde{\xi}(x) = f(x)\zeta(x)$$

for some smooth, nowhere-vanishing function $f\colon S^n \to \mathbb{R}$. Then we must have

$$\widetilde{\xi}(-x) = \widetilde{\xi}(x) \quad \Rightarrow \quad f(-x)\zeta(-x) = f(x)\zeta(x) = f(x)(-\zeta(x)),$$

which implies

$$f(-x) = -f(x).$$

So $f$ must be an odd function. But by the Intermediate Value Theorem, any continuous odd function on $S^n$ must vanish somewhere, contradicting the assumption that $f$ is nowhere vanishing.

Therefore, no such $\mathbb{Z}_2$-invariant, nowhere-vanishing section $\widetilde{\xi}$ exists, and hence the line bundle $\gamma$ is non-trivial. $\square$

**Exercise 11.16.** Thinking of $S^3$ as the level set of the smooth function $f(z, w) = |z|^2 + |w|^2$ on $\mathbb{C}^2$, the tangent bundle $TS^3$ is given by

$$TS^3 = \ker(df = zd\overline{z} + \overline{z}dz + wd\overline{w} + \overline{w}dw)|_{S^3}.$$

The vector fields

$$\begin{aligned}
\xi_1 &= iz\frac{\partial}{\partial z} + iw\frac{\partial}{\partial w} - i\bar{z}\frac{\partial}{\partial \bar{z}} - i\bar{w}\frac{\partial}{\partial \bar{w}},\\
\xi_2 &= -w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial \bar{z}} + \bar{z}\frac{\partial}{\partial \bar{w}},\\
\xi_3 &= iw\frac{\partial}{\partial z} - iz\frac{\partial}{\partial w} - i\bar{w}\frac{\partial}{\partial \bar{z}} + i\bar{z}\frac{\partial}{\partial \bar{w}},
\end{aligned}$$

define a global frame for $TS^3$. Therefore, by Lemma 11.10, $TS^3$ is trivial. $\square$

**Exercise 11.17.** In Exercise 6.12, we proved that the complex vector space of holomorphic sections of $T\mathbb{CP}^1$ is 3-dimensional. On the other hand, a holomorphic section of $\mathbb{CP}^1 \times \mathbb{C}$ is simply a holomorphic function, and the only holomorphic functions on $\mathbb{CP}^1$ are constants. Therefore, the complex vector space of holomorphic sections of $\mathbb{CP}^1 \times \mathbb{C}$ is 1-dimensional. We conclude that $T\mathbb{CP}^1$ and $\mathbb{CP}^1 \times \mathbb{C}$ are not isomorphic. □

**Exercise 11.18.** We have rank $\mathrm{Hom}(E, E') =$ rank $E \times$ rank $E'$ because if rank $E = k$ and rank $E' = k'$, then $\mathrm{Hom}(E_p, E'_p)$ can be identified with $M_{k'\times k}(\mathbb{R})$ or $M_{k'\times k}(\mathbb{C})$ for all $p \in M$.

By Lemma 11.10, a line bundle is trivial if and only if it admits a nowhere vanishing section. By the previous step, if $L$ is a line bundle, then $\mathrm{Hom}(L, L)$ is also a line bundle. Moreover, the identity homomorphism $\mathrm{id}\colon L \longrightarrow L$ defines a nowhere vanishing section of $\mathrm{Hom}(L, L)$.

Chapter 12

# Dual of vector bundles

There are many operations in linear algebra that transform one vector space into another. In general, any such basis-independent operation can also be applied fiber-wise to vector bundles, producing important and interesting new bundles from a given one. In this lecture, we will learn about duals of vector bundles, which leads to the definition of the cotangent bundle and differential 1-forms. We will see more examples in the next lectures.

The **dual** of a vector space $V$ over the ground field $\mathbb{F}$ is the vector space

$$V^* = \mathrm{Hom}(V, \mathbb{F})$$

of linear maps from $V$ to $\mathbb{F}$. The dual of any vector space has the same dimension. Any choice of basis $(e_1, \ldots, e_k)$ on $V$ determines a dual basis $(e_1^*, \ldots, e_k^*)$ on $V^*$ defined by

$$e_i^*(e_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$

These bases determine an isomorphism $V \longrightarrow V^*$ that sends $e_i$ to $e_i^*$. The same construction can be applied (fiber-wise) to vector bundles (producing a co-frame from a frame).

**Lemma 12.1.** *Given a vector bundle $E \longrightarrow M$, there exists a dual vector bundle of the same regularity type, $E^* \longrightarrow M$, such that $E_p^* = (E_p)^*$ for all $p \in M$.*

**Proof.** Given a presentation of $E$ as in (11.2), the dual vector bundle is

$$E^* = \coprod_{\alpha \in \mathcal{I}} (V_\alpha \times F_\alpha^*)/\sim, \quad \text{where } (x,\eta_\alpha) \sim (y,\eta_\beta) \Leftrightarrow \tag{12.1}$$

$$(y,\eta_\beta) \in V_{\beta,\alpha} \times F_\beta^*,\ (x,\eta_\alpha) \in V_{\alpha,\beta} \times F_\alpha^*,\ (x, \Phi_{\alpha\mapsto\beta}(x)^*(\eta_\beta)) = \big(\varphi_{\alpha\mapsto\beta}(x), \eta_\beta\big)$$

Here,

$$\Phi^*_{\alpha\mapsto\beta}(x)\colon F_\beta^* \longrightarrow F_\alpha^*$$

is the dual of $\Phi_{\alpha\mapsto\beta}(x)$, which reverses direction. In general, if

$$L\colon F_1 \longrightarrow F_2$$

is a linear map between two vector spaces, its dual is the linear map

$$L^*\colon F_2^* \longrightarrow F_1^*$$

defined by

$$L^*(\eta)(v) = \eta(L(v)) \qquad \forall\ v \in F_1,\ \eta \in F_2^* = \mathrm{Hom}(F_2, \mathbb{F}).$$

If one prefers the transition maps to go in the usual direction, we instead take the inverse of $\Phi^*_{\alpha\mapsto\beta}(x)$. That is, the transition maps of $E^*$ in the standard sense are $\Phi^*_{\alpha\mapsto\beta}(x)^{-1}$. Since

$$\Phi_{\beta\mapsto\alpha} = \Phi^{-1}_{\alpha\mapsto\beta},$$

we may also write $\Phi^*_{\beta\mapsto\alpha}(\varphi_{\alpha\mapsto\beta}(x))$ instead of $(\Phi^*_{\alpha\mapsto\beta}(x))^{-1}$. When $F_\alpha = \mathbb{R}^k$ or $\mathbb{C}^k$ for all $\alpha \in \mathcal{I}$, the maps $\Phi_{\alpha\mapsto\beta}$ are matrix-valued functions

$$\Phi_{\alpha\mapsto\beta}\colon V_{\alpha,\beta} \longrightarrow \mathrm{GL}(k,\mathbb{R}) \text{ or } \mathrm{GL}(k,\mathbb{C}).$$

By the following exercise, $\Phi^*_{\alpha\mapsto\beta}(x)^{-1}$ corresponds to the transpose inverse of $\Phi_{\alpha\mapsto\beta}(x)$. Thus, the transition maps of $E^*$ inherit the same regularity as those of $E$ (i.e., if $E$ is continuous, smooth, or holomorphic, then so is $E^*$). □

**Exercise 12.2.** For any linear map

$$L\colon \mathbb{F}^m \longrightarrow \mathbb{F}^n, \quad v \mapsto Av,$$

given by an $n \times m$ matrix $A$, show that the matrix of

$$L^*\colon (\mathbb{F}^n)^* \cong \mathbb{F}^n \longrightarrow (\mathbb{F}^m)^* \cong \mathbb{F}^m$$

is $A^T$. Here, the identification $(\mathbb{F}^n)^* \cong \mathbb{F}^n$ uses the standard basis of $\mathbb{F}^n$.

**The example of the cotangent bundle.**

**Definition 12.3.** For every smooth or holomorphic manifold $M$, the dual of the tangent bundle is called the (smooth or holomorphic) cotangent bundle, denoted $T^*M$.

If
$$\varphi = (x_1, \ldots, x_m) \colon U \longrightarrow V \subset \mathbb{R}^n \text{ or } \mathbb{C}^n$$
is a chart on $M$, then $\partial_{x_1}, \ldots, \partial_{x_m}$ form a local frame for $TM$ over $U$, and $dx_1, \ldots, dx_m$ form the dual frame for $T^*M$, satisfying
$$dx_i(\partial_{x_j}) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise.} \end{cases}$$
A section of $TM$ over $U$ is called a vector field; a section of $T^*M$ is called a **differential** 1**-form**. With respect to the co-frame $dx_1, \ldots, dx_m$, every differential 1-form has the local expression
$$\eta = \sum_{i=1}^{m} a_i(x)\ dx_i$$
for some smooth or holomorphic functions $a_i(x)$.

For every smooth function $f \colon U \longrightarrow \mathbb{R}$ (or holomorphic $f \colon U \longrightarrow \mathbb{C}$), the derivative of $f$ at any point $x \in U$ is a linear map
$$d_x f \colon T_x M \longrightarrow T_{f(x)}\mathbb{R} = \mathbb{R}.$$
Therefore, by the definition of the dual bundle, *df* defines a section of $T^*M$ over $U$ (i.e., a differential 1-form), which has the local expression
$$df = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}\ dx_i.$$
For this reason, the derivative of $f$ is often referred to as the differential of $f$. We will later see that every differential 1-form is locally of the form *df* for some smooth or holomorphic function $f$ – this is the content of the Poincaré Lemma.

Globally, recall from (6.3) that, if $M$ is presented as a quilted space
$$M := \coprod_{\alpha \in \mathcal{I}} V_\alpha \big/ \big\{ V_{\alpha,\beta} \ni x \sim \varphi_{\alpha \mapsto \beta}(x) \in V_{\beta,\alpha} \text{ for all } \alpha, \beta \in \mathcal{I} \big\},$$
then a vector field on $M$ is a collection of local vector fields $X_\alpha$ on $V_\alpha$ satisfying the compatibility condition:
$$X_\beta|_{V_{\beta,\alpha}} = d\varphi_{\alpha \mapsto \beta}(X_\alpha|_{V_{\alpha,\beta}}).$$
Since $d\varphi_{\alpha \mapsto \beta}$ pushes $X_\alpha|_{V_{\alpha,\beta}}$ forward by a diffeomorphism to a vector field on an open subset of $V_\beta$, it is often called the **push-forward map**, denoted $(\varphi_{\alpha \mapsto \beta})_*$.

Similarly, by (12.1), a global differential 1-form $\eta$ on $M$ is a collection of local 1-forms $\eta_\alpha$ on $V_\alpha$ satisfying the compatibility condition:
$$\eta_\alpha|_{V_{\beta,\alpha}} = \varphi^*_{\alpha \mapsto \beta}(\eta_\beta|_{V_{\alpha,\beta}}).$$

Here, the **pullback map**

$$\varphi^*_{\alpha\mapsto\beta} := (d\varphi_{\alpha\mapsto\beta})^*$$

is the dual of the push-forward map and acts by composition with $d\varphi_{\alpha\mapsto\beta}$. That is, for a vector field $X_\alpha$, the action is defined by

$$\big(\varphi^*_{\alpha\mapsto\beta}(\eta_\beta)\big)(X_\alpha) := \eta_\beta\big((\varphi_{\alpha\mapsto\beta})_*(X_\alpha)\big) = \eta_\beta\big(d\varphi_{\alpha\mapsto\beta}(X_\alpha)\big).$$

If $x = (x_1, \dots, x_m)$ are coordinates on $V_\alpha$ and $y = (y_1, \dots, y_m)$ are coordinates on $V_\beta$, with $y = \varphi_{\alpha\mapsto\beta}(x)$ on the overlap, then

$$\eta_\beta = \sum_{i=1}^m b_i(y)\ dy_i$$

and

$$\varphi^*_{\alpha\mapsto\beta}(\eta_\beta) = \eta_\beta \circ d\varphi_{\alpha\mapsto\beta} = \sum_{i=1}^m b_i(\varphi_{\alpha\mapsto\beta}(x))\ dy_i \circ d\varphi_{\alpha\mapsto\beta}.$$

By the chain rule,

$$dy_i \circ d\varphi_{\alpha\mapsto\beta} = d(y_i(x)) = \sum_{j=1}^m \frac{\partial y_i}{\partial x_j}\ dx_j.$$

*Therefore, computing pullbacks is straightforward:*

$$\varphi^*_{\alpha\mapsto\beta}\left(\sum_{i=1}^m b_i(y)\ dy_i\right) = \sum_{i=1}^m \sum_{j=1}^m b_i(\varphi_{\alpha\mapsto\beta}(x)) \frac{\partial y_i}{\partial x_j}\ dx_j.$$

If

$$\eta_\alpha = \sum_{i=1}^m a_i(x)\ dx_i,$$

then the compatibility condition reads

$$a_i(x) = \sum_{j=1}^m b_j(\varphi_{\alpha\mapsto\beta}(x)) \frac{\partial y_j}{\partial x_i}$$

on the overlap.

More generally, if $f\colon M \to N$ is a smooth or holomorphic map between two manifolds, the derivative of $f$ is a vector bundle homomorphism

$$df\colon TM \longrightarrow TN$$

lifting $f$. The dual of this is called the **pullback by** $f$ and is a vector bundle homomorphism

$$f^* := (df)^*\colon T^*N \longrightarrow T^*M$$

in the reverse direction. Formally, it is simply composition with $df$; i.e., for every $p \in M$, $v \in T_pM$, and $\eta \in T^*_{f(p)}N$, we have

$$f^*\eta \in T^*_pM, \quad (f^*\eta)(v) = \eta(d_pf(v)).$$

In local coordinates, the definition is the same as above. If $x = (x_1, \ldots, x_m)$ are local coordinates on $U \subset M$ and $y = (y_1, \ldots, y_n)$ are coordinates on $U' \subset N$, with $y = f(x)$ on $f^{-1}(V) \cap U$, then a 1-form $\eta$ on $U'$ has local expansion

$$\eta = \sum_{i=1}^{n} b_i(y)\ dy_i$$

and

$$f^*(\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_i(f(x))\frac{\partial y_i}{\partial x_j}\ dx_j.$$

**Exercise 12.4.** Thinking of $S^1$ as $\mathbb{R}/\mathbb{Z}$, show that the 1-form $dt$ on $\mathbb{R}$ descends to a 1-form on $S^1$. Consider the standard embedding $\iota\colon S^1 \hookrightarrow \mathbb{R}^2$ of $S^1$ in $\mathbb{R}^2$ and find the pullback of differential 1-form

$$\eta = \frac{-xdy + ydx}{x^2 + y^2}$$

on $\mathbb{R}^2 - \{0\}$ to $S^1$ in terms of $dt$.

**Exercise 12.5.** Recall that the 2-sphere $S^2 \subset \mathbb{R}^3$ can be covered by two charts $\varphi_\pm\colon U_\pm \to V_\pm \cong \mathbb{R}^2$, with transition map

$$\varphi_{+\mapsto -}\colon V_{+,-} = \mathbb{R}^2 \setminus \{0\} \to V_{-,+} = \mathbb{R}^2 \setminus \{0\}, \quad x = (x_1, x_2) \mapsto \frac{1}{|x|^2}(x_1, x_2).$$

Does the differential 1-form $x_1dx_1 + x_2dx_2$ on $V_-$ extend smoothly to the entire $S^2$?

## Solutions to exercises

**Exercise 12.2.** In the domain of $L$, $(\mathbb{F}^n)^*$ is identified with $\mathbb{F}^n$ by letting a column vector $w \in \mathbb{F}^n$ act as a linear map $\eta_w$ on $\mathbb{F}^n$ via

$$\eta_w \colon \mathbb{F}^n \longrightarrow \mathbb{F}, \quad v \longrightarrow w^T \cdot v \in \mathbb{F} \qquad \forall\, v \in \mathbb{F}^n.$$

The identification of $(\mathbb{F}^m)^*$ with $\mathbb{F}^m$ on the target is defined similarly.

By definition, for every $w \in \mathbb{F}^m$ we have

$$\begin{aligned}(L^*\eta_w)(v) &= \eta_w(L(v)) = w^T L(v)\\ &= w^T A v = (A^T w)^T v = \eta_{(A^T w)}(v) \quad \forall\, v \in \mathbb{F}^n.\end{aligned}$$

Therefore, under the identifications of $(\mathbb{F}^n)^*$ with $\mathbb{F}^n$ and $(\mathbb{F}^m)^*$ with $\mathbb{F}^m$ as above, $L^*$ maps $w$ to $A^T w$. □

**Exercise 12.4.** The group of integers $\mathbb{Z}$ acts on $\mathbb{R}$ by translations:

$$\varphi_n \colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad \varphi_n(t) = t + n \quad \forall\, n \in \mathbb{Z},\ t \in \mathbb{R}.$$

The 1-form $dt$ is invariant under this action; i.e. $\varphi_n^* dt = d(t+n) = dt$ for every $n \in \mathbb{Z}$. Therefore, it descends to a 1-form in the quotient space.

The standard embedding $\iota \colon S^1 \hookrightarrow \mathbb{R}^2 = \mathbb{C}$ of $S^1$ in $\mathbb{R}^2$ is given by

$$[t] \longrightarrow \mathrm{e}^{2\pi \mathfrak{i} t} = \cos(2\pi t) + \mathfrak{i} \sin(2\pi t).$$

Therefore,

$$\begin{aligned}\iota^* \left( \frac{-x dy + y dx}{x^2 + y^2} \right) &= \frac{-\cos(2\pi t) d \sin(2\pi t) + \sin(2\pi t) d \cos(2\pi t)}{1}\\ &= 2\pi \left\{ -\cos(2\pi t)^2 dt - \sin(2\pi t)^2 dt \right\} = -2\pi dt.\end{aligned}$$

□

**Exercise 12.5.** For the 1-form $\eta$ in the question, we need to find $\varphi_{+\mapsto -}^* \eta$ on $V_{+,-} = \mathbb{R}^2 - \{0\}$ and check whether the resulting expression extends to the origin. We have

$$\begin{aligned}\varphi_{+\mapsto -}^*(x_1 dx_1 + x_2 dx_2) &= (x_1/|x|^2)\, d(x_1/|x|^2) + (x_2/|x|^2)\, d(x_2/|x|^2)\\ &= (x_1/|x|^6)\,((x_2^2 - x_1^2)\, dx_1 - 2x_1 x_2\, dx_2)\\ &\quad + (x_2/|x|^6)\,((x_1^2 - x_2^2)\, dx_2 - 2x_1 x_2\, dx_1)\\ &= -\frac{x_1 dx_1 + x_2 dx_2}{|x|^4}.\end{aligned}$$

The last expression clearly does not extend to the origin.

Chapter 13

# Pullback, direct sum, and quotient

In general, a vector bundle homomorphism is a commutative diagram

$$\begin{tikzcd} E \arrow[r, "h"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

lifting a map between two manifolds. However, in many discussions, it is more convenient if the underlying map is the identity map $\mathrm{id}_M \colon M \longrightarrow M$. The following enables reducing any vector bundle homomorphism to one over $\mathrm{id}_M \colon M \longrightarrow M$.

**Lemma 13.1.** *Given a continuous, smooth, or holomorphic vector bundle $E' \longrightarrow M'$ and a map $f \colon M \longrightarrow M'$ of the same regularity type, there exists a so-called* **pullback** *vector bundle $f^*E' \longrightarrow M$ of the same regularity type such that*

- $(f^*E')_p = E'_{f(p)}$ *for all $p \in M$,*
- *$f^*E'$ admits a canonical vector bundle homomorphism $f^*E' \longrightarrow E'$ lifting $f$ of the same regularity type;*
- *every vector bundle homomorphism $h \colon E \longrightarrow E'$ lifting $f$ factors through $f^*E'$ in the following sense:*

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[rr, bend left, "h"] & f^*E' \arrow[r] \arrow[d] & E' \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"] \arrow[rr, bend right, "f"'] & V_\beta \arrow[r, "f"] & M' \end{tikzcd}$$

By abuse of notation, we will also denote the vector bundle homomorphism $E \longrightarrow f^*E'$ by $h$.

**Proof.** Fix a collection of local trivializations

$$\{\Phi'_\alpha \colon E'|_{U'_\alpha} \longrightarrow U'_\alpha \times F_\alpha\}_{\alpha\in\mathcal{I}}$$

such that $\{U'_\alpha\}_{\alpha\in\mathcal{I}}$ is an open covering of $M'$ and the change of trivialization maps

$$\Phi'_{\alpha\mapsto\beta} \colon U'_\alpha \cap U'_\beta \longrightarrow \mathrm{Isom}(F_\alpha, F_\beta)$$

are continuous, smooth, or holomorphic, depending on the context. Let $U_\alpha = f^{-1}(U'_\alpha)$ and define

$$\Phi_\alpha \colon (f^*E')|_{U_\alpha} \longrightarrow U_\alpha \times F_\alpha, \quad \Phi_\alpha|_p(v) = \Phi'_\alpha|_{f(p)}(v) \quad \forall v \in (f^*E')_p = E'_{f(p)}.$$

Then, the change of trivialization maps $\Phi_{\alpha\mapsto\beta}$ of the induced collection $\{\Phi_\alpha\}$ are

$$\Phi_{\alpha\mapsto\beta} = \Phi'_{\alpha\mapsto\beta} \circ f \colon U_\alpha \cap U_\beta \longrightarrow \mathrm{Isom}(F_\alpha, F_\beta).$$

We conclude that the change of trivialization maps $\Phi_{\alpha\mapsto\beta}$ are also continuous, smooth, or holomorphic, depending on the context.

The canonical map $f^*E' \longrightarrow E'$ is simply the identity map

$$(f^*E')_p = E'_{f(p)} \xrightarrow{\mathrm{id}_{E'_{f(p)}}} E'_{f(p)}$$

on each fiber. The induced map $h \colon E \longrightarrow f^*E'$ also views $h(v) \in E'_{f(p)}$ as a vector in $(f^*E')_p$ for all $p \in M$ and $v \in E_p$. $\square$

**Remark 13.2.** In light of the previous lemma, given a smooth or holomorphic map $f \colon M \longrightarrow M'$, the derivative of $f$ is often considered as a vector bundle homomorphism

$$df \colon TM \longrightarrow f^*TM'$$

over the identity map on $M$.

For every two vector bundles $E$ and $E'$ over the same base $M$, the **direct sum** $E \oplus E' \longrightarrow M$ of $E$ and $E'$ is the vector bundle with fibers $E_p \oplus E'_p$ at every point $p$. Direct sum of local trivializations for $E$ and $E'$ define local trivialization of the direct sum bundle. With notation as in (11.1) and

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, if the change of trivialization maps $\Phi_{\alpha\mapsto\beta}$ for $E$ and $\Phi'_{\alpha\mapsto\beta}$ for $E'$ are realized as matrix-valued functions

$$\Phi_{\alpha\mapsto\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(k,\mathbb{F}) \quad \text{and} \quad \Phi'_{\alpha\mapsto\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(k',\mathbb{F}),$$

then the change of trivialization matrix-valued functions corresponding to $E \oplus E'$ are the block-diagonal matrices

$$\begin{bmatrix} \Phi_{\alpha\mapsto\beta} & 0 \\ 0 & \Phi'_{\alpha\mapsto\beta} \end{bmatrix}. \tag{13.1}$$

The direct sum bundle $E \oplus E'$ sits as the middle term of a short exact sequence of vector bundles over $M$

$$0 \longrightarrow E \longrightarrow E \oplus E' \longrightarrow E' \longrightarrow 0,$$

where $E \longrightarrow E \oplus E'$ is the vector bundle embedding $v \longrightarrow v \oplus 0$, for all $v \in E$, and the vector bundle quotient map $E \oplus E' \longrightarrow E'$ is simply projection to the second factor. This is a special case of an arbitrary short exact sequence of vector bundles that can be associated to any vector bundle embedding as follows.

**Lemma 13.3.** *Suppose $E$ and $E''$ are continuous, smooth, or holomorphic vector bundles on $M$ and $\iota\colon E \longrightarrow E''$ is a vector bundle embedding (of the same regularity type) over (the identity map of) $M$. Then there exists a "quotient" vector bundle $E' = E''/E$ of the same regularity type such that $E'_p$ is the quotient vector space $E''_p/E_p$ for all $p \in M$, leading to a short exact sequence of vector bundles*

$$0 \longrightarrow E \longrightarrow E'' \longrightarrow E' \longrightarrow 0.$$

**Proof.** With $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, if rank $E'' = r''$ and rank $E = r$, then by the solution to Exercise 11.14, there is an open covering $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ of $M$ and local trivializations

$$\Phi''_\alpha\colon E''|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{F}^{r''}$$

such that the restriction of $\Phi''_\alpha$ to $E|_{U_\alpha}$ also defines a local trivialization

$$\Phi_\alpha := \Phi''_\alpha|_{E|_{U_\alpha}}\colon E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{F}^r.$$

We conclude that the change of trivialization matrix-valued functions of $E$ and $E''$ are related by

$$\Phi''_{\alpha\mapsto\beta} = \begin{bmatrix} \Phi_{\alpha\mapsto\beta} & * \\ 0 & \Phi'_{\alpha\mapsto\beta} \end{bmatrix}, \tag{13.2}$$

for some $(r'' - r) \times (r'' - r)$ matrix block $\Phi'_{\alpha\mapsto\beta}$ in the lower right position. If $\Phi''_{\alpha\mapsto\beta}$ is continuous, smooth, or holomorphic, the same holds for $\Phi'_{\alpha\mapsto\beta}$.

With $r' := r'' - r$, using the canonical decomposition $\mathbb{F}^{r''} = \mathbb{F}^r \oplus \mathbb{F}^{r'}$, and taking the quotient of $\Phi''_\alpha$ and $\Phi_\alpha$, we obtain local trivializations

$$\Phi'_\alpha \colon (E''/E)|_{U_\alpha} \longrightarrow U_\alpha \times (\mathbb{F}^{r''}/\mathbb{F}^r) = U_\alpha \times \mathbb{F}^{r'}$$

such that the change of trivialization matrix-valued functions are $\Phi'_{\alpha\mapsto\beta}$. $\square$

**Example 13.4.** Suppose $f \colon M \longrightarrow M'$ is an immersion. Then, from the point of view of Remark 13.2,

$$df \colon TM \longrightarrow f^*TM'$$

is a vector bundle embedding. The **normal bundle** of $f$ is the quotient bundle

$$\mathcal{N}f = \frac{f^*TM'}{TM},$$

which fits into a short exact sequence of vector bundles

$$0 \longrightarrow TM \longrightarrow f^*TM' \longrightarrow \mathcal{N}f \longrightarrow 0.$$

For submanifolds $M \subset M'$, i.e., when $f$ is simply the inclusion map of a submanifold, we will denote the normal bundle of $M$ in $M'$ by $\mathcal{N}_{M'}M$ instead.

**Exercise 13.5.** Suppose $f \colon M \longrightarrow M'$ is a smooth or holomorphic map, and let $Y = f^{-1}(q) \subset M$ be a regular level set (thus, a submanifold). Show that $\mathcal{N}_M Y$ is trivial.

**Exercise 13.6.** Here is a generalization of the previous exercise. Suppose $f \colon M \longrightarrow M'$ is a smooth or holomorphic map which is transverse to $Y' \subset M'$. Let $Y = f^{-1}(Y') \subset M$. Show that $\mathcal{N}_M Y = (f|_Y)^* \mathcal{N}_{M'} Y'$.

**Definition 13.7.** We say a short exact sequence of vector bundles

$$0 \longrightarrow E \longrightarrow E'' \longrightarrow E' \longrightarrow 0.$$

**splits** if there is an isomorphism $E'' \cong E \oplus E'$ compatible with the inclusion and projections above; i.e. the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & E'' \arrow[r] \arrow[d] & E' \arrow[d] \\
E \arrow[r] & E \oplus E' \arrow[r] & E'
\end{tikzcd}$$

**Exercise 13.8.** Show that every continuous or smooth short exact sequence of vector bundles splits.

**Remark 13.9.** Pullback commutes with all pointwise operations on vector bundles; for example, the pullback of a dual bundle or a direct sum is the dual or direct sum of the pullbacks, respectively. Moreover, if $f \colon M \longrightarrow M'$ is a continuous, smooth, or holomorphic map and $s \colon M' \longrightarrow E'$ is a continuous, smooth, or holomorphic section of $E'$, then there is a pullback section $f^*s$ of the same regularity defined by

$$(f^*s)(p) = s(f(p)) \in E'_{f(p)} = (f^*E')_p \qquad \forall\, p \in M.$$

Therefore, pullback induces a module homomorphism from the $C^0(M')$, $C^\infty(M')$, or $C^{\mathrm{hol}}(M')$-module of sections of $E'$ to the corresponding $C^0(M)$, $C^\infty(M)$, or $C^{\mathrm{hol}}(M)$-module of sections of $f^*E'$, covering the algebra homomorphism between functions on $M'$ and functions on $M$.

**Exercise 13.10.** Suppose $\pi\colon E \longrightarrow M$ is a smooth or holomorphic vector bundle. Considering $E$ as a manifold, show that the tangent bundle $TE$ of $E$ fits into a long exact sequence

$$0 \longrightarrow \pi^*E \longrightarrow TE \xrightarrow{d\pi} \pi^*TM \longrightarrow 0.$$

If $E$ is a smooth vector bundle, use Exercise 13.8 to conclude that $TE \cong \pi^*(E \oplus TM)$.

## Solutions to exercises

**Exercise 13.5.** By definition, for every $p \in Y$,

$$d_p f\colon TM \longrightarrow T_q M'$$

is surjective and $T_p Y = \ker(d_p f)$. Therefore, $df$ induces an isomorphism

$$df|_Y\colon \mathcal{N}_M Y = \frac{TM|_Y}{TY} \longrightarrow Y \times T_q M'.$$

This is an isomorphism between the normal bundle $\mathcal{N}_M Y$ and the product vector bundle $Y \times T_q M'$ (note that $T_q M'$ is a fixed vector space). $\square$

**Exercise 13.6.** Generalizing the previous proof, by definition of transversality, for every $p \in Y$, the composition

$$\mathrm{pr} \circ d_p f\colon TM \longrightarrow T_{f(p)} M'/T_{f(p)} Y'$$

is surjective and $T_p Y = \ker(\mathrm{pr} \circ d_p f)$, where

$$\mathrm{pr}\colon TM'|_Y \longrightarrow \mathcal{N}_{M'} Y' = \frac{TM'|_Y}{TY}$$

is the quotient projection map. Therefore, $df$ induces an isomorphism

$$\mathrm{pr} \circ df|_Y\colon \mathcal{N}_M Y = \frac{TM|_Y}{TY} \longrightarrow (f|_Y)^* \mathcal{N}_{M'} Y'.$$

$\square$

**Exercise 13.8.** There is a relatively easy way to prove this using metrics on vector bundles; however, both the following argument and the proof of existence of metrics – as we will see soon – rely on partition of unity. Holomorphic manifolds do not admit partition of unity. Therefore, the proof does not extend to holomorphic vector bundles. In fact, not every holomorphic short exact sequence of vector bundles splits and there are "cohomological" obstructions that are related to the $*$ component in (13.2).

There is a one-to-one correspondence between splittings $E'' = E \oplus E'$ and embeddings $\iota\colon E' \longrightarrow E''$ whose composition with the projection map $\mathrm{pr}\colon E'' \longrightarrow E'$ is $\mathrm{id}_{E'}$. We construct such an embedding. By Exercise 11.14, over a sufficiently small neighborhood $U$ of any point in $M$, a local frame $(s_1, \ldots, s_k)$ for $E$ extends to a local frame $(s_1, \ldots, s_{k''})$ for $E''$. Therefore, restricted to the subspace $\langle s_{k+1}, \ldots, s_{k''} \rangle \subset E''|_U$, the projection map

$$\langle s_{k+1}, \ldots, s_{k''} \rangle \longrightarrow E'|_U$$

is an isomorphism, The inverse of this map gives an embedding

$$\iota_U\colon E'|_U \longrightarrow E''|_U$$

whose composition with the projection map $\mathrm{pr}\colon E''|_U \longrightarrow E'_U$ is the identity map.

Fix an open covering $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ of $M$ and local embeddings (as constructed above)

$$\iota_\alpha\colon E'|_{U_\alpha} \longrightarrow E''|_{U_\alpha},$$

as well as a partition of unity $\{\varrho_\alpha\colon U_\alpha \longrightarrow [0,1]\}_{\alpha\in\mathcal{I}}$ subordinate to this covering. For each $\alpha$, the product

$$\varrho_\alpha\iota_\alpha\colon E'|_{U_\alpha} \longrightarrow E''|_{U_\alpha}$$

extends a similarly denoted vector bundle homomorphism

$$\varrho_\alpha\iota_\alpha\colon E' \longrightarrow E''$$

on the entire $M$ that is trivial (i.e. zero) homomorphism outside $U_\alpha$. Let

$$\iota = \sum_\alpha \varrho_\alpha\iota_\alpha\colon E' \longrightarrow E''.$$

Here, we are adding a countable collection of homomorphisms – that is, a countable collection of sections of $\mathrm{Hom}(E', E'')$ – such that the sum is finite in a neighborhood of each point. For every $v \in E''$, we have

$$\mathrm{pr}(\iota(v)) = \sum_\alpha \mathrm{pr}(\varrho_\alpha\iota_\alpha(v)) = \sum_{\alpha\colon v\in E|_{U_\alpha}} \varrho_\alpha \mathrm{pr}(\iota_\alpha(v)) = \Big(\sum_{\alpha\colon v\in E|_{U_\alpha}} \varrho_\alpha\Big)v = v.$$

□

**Exercise 13.10.** For every $p \in E$ and $v \in E_p$, the kernel of the derivative map

$$d_v\pi\colon T_vE \longrightarrow T_pM$$

is the set of vectors tangent to the fiber $E_p$ at $v$; that is, $\ker(d_v\pi) = T_vE_p$. Since $E_p$ is a vector space, using parallel transport, the tangent space at any $v \in E_p$ is canonically identified with $T_0E_p = E_p$. Therefore, there is a canonical isomorphism

$$\ker(d\pi)|_v \cong E_{\pi(v)} \qquad \forall\ v \in E.$$

By the definition of pullback, this implies that

$$\ker(d\pi) = \pi^*E.$$

Furthermore, from the perspective of Remark 13.2, the map $d\pi$ is surjective onto $\pi^*TM$. We conclude that $TE$ fits into a short exact sequence

$$0 \longrightarrow \pi^*E \longrightarrow TE \xrightarrow{d\pi} \pi^*TM \longrightarrow 0.$$

By Exercise 13.8, every short exact sequence of smooth vector bundles splits. Since pullback and direct sum commute, we obtain

$$TE \cong \pi^*E \oplus \pi^*TM = \pi^*(E \oplus TM).$$

$\square$

Chapter 14

# Tensor and exterior products

Suppose $V_1, \ldots, V_k$ and $W$ are vector spaces over a field $\mathbb{F}$. We say that

$$L\colon V_1 \times \cdots \times V_k \longrightarrow W$$

is a **$k$-linear** map if it is linear in each input. The concept of the tensor product, discussed below, allows us to realize $L$ as a linear map defined on a vector space other than the product $V_1 \times \cdots \times V_k$ itself. More precisely, the **tensor product** $V_1 \otimes \cdots \otimes V_k$ is the vector space generated by $k$-tuples of vectors $(v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$, presented as $v_1 \otimes \cdots \otimes v_k$, subject to the following relations:

- $v_1 \otimes \cdots \otimes (v_i + v_i') \otimes \cdots \otimes v_k = (v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k) + (v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_k)$;
- $c(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k) = v_1 \otimes \cdots \otimes (cv_i) \otimes \cdots \otimes v_k$ for all $i = 1, \ldots, k$ and $c \in \mathbb{F}$.

It is a classical result in linear algebra that every $k$-linear map $L\colon V_1 \times \cdots \times V_k \longrightarrow W$ factors through $V_1 \otimes \cdots \otimes V_k$; that is, $L$ induces a linear map

$$\widetilde{L}\colon V_1 \otimes \cdots \otimes V_k \longrightarrow W$$

such that $L$ is the composition of $\widetilde{L}$ with the natural ($k$-linear) product map

$$V_1 \times \cdots \times V_k \longrightarrow V_1 \otimes \cdots \otimes V_k, \qquad (v_1, \ldots, v_k) \mapsto v_1 \otimes \cdots \otimes v_k.$$

For each $i = 1, \ldots, k$, suppose $\{e_{i,1}, \ldots, e_{i,m_i}\}$ is a basis of $V_i$. Then the collection

$$\{e_{1,j_1} \otimes \cdots \otimes e_{k,j_k}\}_{j_1,\ldots,j_k}$$

is a basis for $V_1 \otimes \cdots \otimes V_k$. In particular,

$$\dim(V_1 \otimes \cdots \otimes V_k) = \prod_{i=1}^{k} \dim V_i.$$

It is easy to see that tensor product is associative; i.e.

$$V_1 \otimes V_2 \otimes V_3 = (V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3).$$

A collection of linear maps

$$L_i \colon V_i \longrightarrow W_i, \quad i = 1, \ldots, \ell,$$

induces a linear map

$$L = L_1 \otimes \cdots \otimes L_\ell \colon V_1 \otimes \cdots \otimes V_\ell \longrightarrow W_1 \otimes \cdots \otimes W_k$$

that sends $v_1 \otimes \cdots \otimes v_\ell$ to $L_1(v_1) \otimes \cdots \otimes L_\ell(v_\ell)$. Also, taking dual and pullback commute with tensor product, and

$$(V_1 \oplus V_2) \otimes V_3 = (V_1 \otimes V_3) \oplus (V_2 \otimes V_3).$$

Similarly to the other operations discussed in previous sections, this definition extends point-wise to a collection of vector bundles over the same manifold $M$. Furthermore, the identities above hold for vector bundles as well.

**Lemma 14.1.** *Given continuous, smooth, or holomorphic vector bundles $E_1, \ldots, E_k \longrightarrow M$, there exists a so-called* **tensor product** *vector bundle*

$$E_1 \otimes \cdots \otimes E_k \longrightarrow M$$

*of the same regularity type such that*

$$(E_1 \otimes \cdots \otimes E_k)_p = (E_1)_p \otimes \cdots \otimes (E_k)_p \quad \forall\ p \in M.$$

**Proof.** With $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, fix an open covering $\{U_\alpha\}_{\alpha \in \mathcal{I}}$ of $M$ and local trivializations

$$\Phi_{i,\alpha} \colon E_i|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{F}^{r_i}$$

with the corresponding change-of-trivialization matrix-valued functions

$$\Phi_{i,\alpha\mapsto\beta} \colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(r_i, \mathbb{F})$$

that are continuous, smooth, or holomorphic, depending on the context.

The tensor product of the vector bundle homomorphisms $\Phi_{i,\alpha}$ defines local trivializations

$$\Phi_{1,\alpha} \otimes \cdots \otimes \Phi_{k,\alpha} \colon (E_1 \otimes \cdots \otimes E_k)|_{U_\alpha} \longrightarrow U_\alpha \times (\mathbb{F}^{r_1} \otimes \cdots \otimes \mathbb{F}^{r_k}) \cong U_\alpha \times \mathbb{F}^{r_1 \cdots r_k}.$$

The change-of-trivialization matrix-valued functions $\Phi_{\alpha\mapsto\beta}$ of these induced trivializations are matrices whose entries are products of the corresponding entries of the $\Phi_{i,\alpha\mapsto\beta}$. Therefore, the tensor product bundle $E_1 \otimes \cdots \otimes E_k$ has the same regularity type as its constituents. $\square$

**Example 14.2.** For every pair of vector bundles $E, E' \longrightarrow M$, the vector bundle

$$\mathrm{Hom}(E, E') \longrightarrow M$$

of vector bundle homomorphisms (over the identity map of $M$), defined before Exercise 11.18, is canonically isomorphic to $E^* \otimes E'$. To see this, since both are defined point-wise, it suffices to canonically identify the two at the level of vector spaces.

Suppose $V$ and $V'$ are vector spaces and

$$\alpha = \sum_{i=1}^{\ell} \eta_i \otimes v_i' \in V^* \otimes V'.$$

We can interpret $\alpha$ as a linear map $L_\alpha \colon V \longrightarrow V'$ defined by

$$L_\alpha(v) = \sum_{i=1}^{\ell} \eta_i(v)\, v_i'.$$

Conversely, suppose $L \colon V \longrightarrow V'$ is a linear map. Fix a basis $e_1, \dots, e_k$ for $V$ and a basis $e_1', \dots, e_\ell'$ for $V'$, and let $e_1^*, \dots, e_k^*$ denote the dual basis of $V^*$. Suppose

$$L(e_i) = \sum_j a_{ij} e_j' \qquad \forall\ i = 1, \dots, k.$$

Then define

$$\alpha = \sum_i \sum_j a_{ij}\, e_i^* \otimes e_j' \in V^* \otimes V'.$$

It is easy to verify that $L = L_\alpha$. We leave it as an exercise to the reader to check that the element $\alpha$ associated to $L$ is independent of the choice of bases.

**Exercise 14.3.** For every vector bundle $E \to M$, show that the vector bundle $E \otimes E^*$ has a nowhere-vanishing section. In particular, show that the tensor product of any line bundle with its dual is naturally isomorphic to the trivial line bundle.

**Remark 14.4.** The set of real or complex line bundles of any regularity type over a manifold forms a group under the operation $\otimes$, with the identity element being the trivial line bundle and the inverse of any element given by its dual line bundle. In the case of complex line bundles, this group is known as the **Picard group**.

In the definition of tensor product, if all vector spaces $V_i$ are the same vector space $V$, we may impose additional symmetry or anti-symmetry relations among the generators as follows.

There are two ways to define the $k$-th symmetric tensor product of a vector space.

**Definition 14.5.** (symmetric tensor product as a quotient space) The **$k$-th symmetric tensor product** $\mathrm{Sym}^k(V)$ of $V$ is the quotient of

$$V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$$

by the symmetry relations

$$v_1 \otimes \cdots \otimes v_k \sim v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \qquad \forall\ \sigma \in S_k,$$

where $S_k$ is the symmetric group on $k$ letters.

From this perspective, we will continue denoting the equivalence class of $v_1 \otimes \cdots \otimes v_k$ in $\mathrm{Sym}^k(V) = V^{\otimes k}/\sim$ by the same expression, bearing in mind that permuting the factors does not change the element in the vector space.

**Definition 14.6.** (symmetric tensor product as a subspace) The **$k$-th symmetric tensor product** $\mathrm{Sym}^k(V)$ of $V$ is the subspace of elements in $V^{\otimes k}$ that are invariant under the permutation action of $S_k$.

From this perspective, an element of $\mathrm{Sym}^k(V)$ is a linear combination in $V^{\otimes k}$ that is symmetric with respect to the group action. For instance, for every $v, v' \in V$, $v \otimes v' + v' \otimes v$ defines an element of $\mathrm{Sym}^2(V)$. We will use the second perspective when presenting metrics in the future sections.

Over a field of characteristic zero such as $\mathbb{R}$ or $\mathbb{C}$, there is a canonical isomorphism between the quotient space (Definition 1) and the subspace (Definition 2), established via the *symmetrization map*:

$$\mathrm{sym}\colon V^{\otimes k} \longrightarrow \mathrm{Sym}^k(V)_{\text{subspace}}, \quad T \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \cdot T.$$

This map descends to an isomorphism

$$V^{\otimes k}/\sim\ \xrightarrow{\sim} \mathrm{Sym}^k(V)_{\text{subspace}} \subset V^{\otimes k}.$$

That is, every class in the quotient has a unique symmetric representative.

Symmetric tensor products of vector bundles are defined (pointwise) in the same way.

**Example 14.7.** An important example of a symmetric tensor is a Riemannian metric on a real vector space/bundle. Given a real vector space $V$, a **Riemannian metric** $g$ on $V$ is a symmetric bilinear map

$$g\colon V \times V \longrightarrow \mathbb{R}$$

that is positive-definite in the sense that

$$g(v, v) > 0 \qquad \forall\ v \neq 0.$$

In other words, $g$ can be regarded as a linear map $g\colon \mathrm{Sym}^2(V) \longrightarrow \mathbb{R}$ satisfying the positive-definiteness condition. Equivalently, $g$ is an element of the dual bundle $\mathrm{Sym}^2(V)^* = \mathrm{Sym}^2(V^*)$ that satisfies the additional positivity condition above.

A continuous or smooth Riemannian metric on a vector bundle $E$ is a continuously or smoothly varying family of Riemannian metrics on fibers of $E$. In other words, $g$ is a continuous or smooth section of $\mathrm{Sym}^2(E^*)$ that is positive-definite on each fiber. The semi-positivity condition is an open condition; that is, if $g$ is a Riemannian metric, then any sufficiently small deformation of it in the space of sections of $\mathrm{Sym}^2(E^*)$ will also be a Riemannian metric. A Riemannian metric on a smooth manifold $M$ is, by definition, a Riemannian metric on its tangent bundle. We will learn more about Riemannian manifolds in upcoming lectures.

Next, we define and study a skew-symmetric analogue of the tensor product.

**Definition 14.8.** The **$k$-th exterior product** $\Lambda^k V$ of $V$ is the quotient of

$$V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$$

by the skew-symmetry relations

$$v_1 \otimes \cdots \otimes v_k \sim \varepsilon(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \qquad \forall\, \sigma \in S_k,$$

where $\varepsilon(\sigma) \in \{\pm 1\}$ is the sign of the permutation $\sigma$. If $\sigma$ is a product of an odd number of transpositions, then $\varepsilon(\sigma) = -1$; otherwise, it is $+1$.

We will denote the equivalence class of $v_1 \otimes \cdots \otimes v_k$ in $\Lambda^k V = V^{\otimes k}/\sim$ by

$$v_1 \wedge \cdots \wedge v_k,$$

bearing in mind that permuting the factors may introduce a sign. For instance, for every $1 \leq i < j \leq k$, we have

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k.$$

Therefore, if $v_i = v_j$ for some $i < j$, then

$$v_1 \wedge \cdots \wedge v_k = 0. \tag{14.1}$$

More generally, successive applications of this fact show that if $v_1, \ldots, v_k$ are linearly dependent, then

$$v_1 \wedge \cdots \wedge v_k = 0.$$

*In fact, for the following reason, $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $v_1, \ldots, v_k$ are linearly dependent.*

If $e_1, \ldots, e_m$ is a frame (ordered basis) for $V$, then the collection

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{0 < i_1 < i_2 < \cdots < i_k \leq m} \tag{14.2}$$

is a basis for $\Lambda^k V$. In particular, $\Lambda^k V$ only makes sense for $0 \leq k \leq m$ and is 0 (i.e., trivial) for other values of $k$.

- For $k = 0$, $\Lambda^0 V$ is defined to be $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, depending on the context.
- For $k = 1$, $\Lambda^1 V$ is simply $V$ itself.
- For $k = \dim V$, $\Lambda^{\mathrm{top}} V := \Lambda^{\dim V} V$ is a 1-dimensional vector space generated by any element of the form

$$\omega = v_1 \wedge \cdots \wedge v_k,$$

where $v_1, \ldots, v_k$ is an ordered basis (or frame) for $V$. Note that any 1-dimensional vector space over the ground field $\mathbb{F}$ can be identified with $\mathbb{F}$, but the identification is not canonical.

A linear map $L\colon V \longrightarrow W$ induces linear maps

$$\Lambda^k L\colon \Lambda^k V \longrightarrow \Lambda^k W, \qquad \forall\ k \geq 0.$$

**Exercise 14.9.** Suppose $L\colon \mathbb{F}^m \longrightarrow \mathbb{F}^m$ is a linear map given by an $m \times m$ matrix $A = [a_{ij}]$. What is the linear map

$$\Lambda^m L\colon \Lambda^m \mathbb{F}^m = \mathbb{F} \longrightarrow \mathbb{F}$$

in terms of $a_{ij}$?

**Exercise 14.10.** Consider the linear map

$$L\colon M_{2\times 2}(\mathbb{F}) \longrightarrow M_{2\times 2}(\mathbb{F}), \quad A \longrightarrow BA$$

where $M_{2\times 2}(\mathbb{F})$ is the space of real $2 \times 2$ matrices and $B \in M_{2\times 2}(\mathbb{F})$. Show that

$$\Lambda^4 L\colon \Lambda^4 M_{2\times 2}(\mathbb{F}) \longrightarrow \Lambda^4 M_{2\times 2}(\mathbb{F})$$

is multiplication by $\det(B)^2$.

Exterior products of vector bundles are defined (pointwise) in the same way.

**Lemma 14.11.** *Given a continuous, smooth, or holomorphic vector bundle $E \longrightarrow M$, for every $0 \leq k \leq \operatorname{rank} E$, there exists a so-called* **exterior product** *vector bundle*

$$\Lambda^k E \longrightarrow M$$

*of the same regularity type such that*

$$(\Lambda^k E)_p = \Lambda^k E_p \qquad \forall\ p \in M.$$

*In particular, $\Lambda^0 E = M \times \mathbb{R}$ or $M \times \mathbb{C}$ is defined to be the trivial bundle, $\Lambda^1 E = E$, and*

$$\Lambda^{\mathrm{top}} E := \Lambda^{\operatorname{rank} E} E \longrightarrow M$$

*is a line bundle whose fiber at any point $p$ is generated by the wedge product of the vectors in a frame for $E_p$.*

**Proof.** The proof of the first statement is identical to that of Lemma 14.1; that is, the change-of-trivialization matrix-valued functions of each $\Lambda^k E$ are matrices whose entries are products of $k \times k$ minors of the change-of-trivialization matrices $\Phi_{\alpha\mapsto\beta}$ of $E$. Therefore, $\Lambda^k E$ has the same regularity type as its constituents. In particular, it follows from the solution to Exercise 14.9 that the change-of-trivialization functions of the line bundle $\Lambda^{\mathrm{top}} E$ are

$$\det(\Phi_{\alpha\mapsto\beta})\colon U_\alpha \cap U_\beta \longrightarrow \mathbb{F}^*.$$

The statements about the special cases are the vector bundle analogues of the items listed before Exercise 17.1. More precisely, for $k = 0$, we define $\Lambda^0 E = M \times \mathbb{F}$. We will later see that this convention is consistent with other constructions. For $k = 1$, $\Lambda^1 E = E$ by definition. For $k = r \coloneqq \operatorname{rank} E$, any local trivialization $E|_U \longrightarrow U \times \mathbb{F}^r$ corresponds to a frame $s_1, \ldots, s_r$ for $E_U$. Wedging these sections defines a nonzero section $\omega = s_1 \wedge \cdots \wedge s_r$ of $(\Lambda^{\mathrm{top}} E)|_U$ (and thus a local trivialization $(\Lambda^{\mathrm{top}} E)|_U \longrightarrow U \times \mathbb{F}$). Changing the local trivialization corresponds to changing the given frame to another frame $s'_1, \ldots, s'_r$. If

$$s'_i = \sum_{j=1}^{r} a_{ij} s_j \qquad \forall\ i = 1, \ldots, r,$$

then

$$\omega' = s'_1 \wedge \cdots \wedge s'_r$$

is related to $\omega$ by

$$\omega' = \det(a_{ij})\, \omega. \tag{14.3}$$

$\square$

**Exercise 14.12.** What is the rank of $\Lambda^k E$?

In the next lecture, we will dive deeper into the definition of the top exterior product and orientability.

## Solutions to exercises

**Exercise 14.3.** A nowhere vanishing section of $E \otimes E^*$ is equivalent to an embedding of the trivial line bundle

$$\mathcal{O} := M \times \mathbb{F}$$

into $E \otimes E^*$. Therefore, we need to find a (canonical) embedding

$$\iota \colon \mathcal{O} \longrightarrow E \otimes E^*$$

of $\mathcal{O}$ into $E \otimes E^*$. Taking duals, this is equivalent to a surjective vector bundle homomorphism (over the identity map of $M$)

$$\iota^* \colon (E \otimes E^*)^* = E^* \otimes E \longrightarrow \mathcal{O}^* = \mathcal{O}.$$

However, the natural pairing between elements of $E$ and those of the dual bundle $E^*$ defines a canonical surjective bundle homomorphism

$$E^* \otimes E \longrightarrow \mathcal{O},$$

which completes the proof of the first statement.

If $E$ is a line bundle, then $E \otimes E^*$ is also a line bundle. Therefore, the above (surjective) morphism is an isomorphism. □

**Exercise 14.9.** Let $e_1, \ldots, e_m$ denote the standard basis of $\mathbb{F}^m$. We have

$$v_j := L(e_j) = \sum_{i=1}^{m} a_{ij} e_j \qquad \forall\ j = 1, \ldots, m.$$

Therefore, the induced map $\Lambda^m L$ is given by

$$e_1 \wedge \cdots \wedge e_m \mapsto v_1 \wedge \cdots \wedge v_m$$

Expanding the wedge product on right we get $m^2$ terms of the form

$$a_{1j_1} a_{2j_2} \cdots a_{mj_m} e_{j_1} \wedge \cdots \wedge e_{mj_m}$$

where $(j_1, \ldots, j_m) \in \{1, \ldots, m\}^m$. However, by (14.1), the latter is nonzero if and only if

$$\sigma = (j_1, \ldots, j_m)$$

is a permutation of $(1, 2, \ldots, m)$. Moreover, if $\sigma \in S_m$, then

$$e_{j_1} \wedge \cdots \wedge e_{mj_m} = \varepsilon(\sigma) e_1 \wedge \cdots \wedge e_m.$$

We conclude that

$$\begin{aligned}(\Lambda^m L)(e_1 \wedge \cdots \wedge e_m) &= \Big( \sum_{\sigma \in S_m} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{m\sigma(m)} \Big)\ e_1 \wedge \cdots \wedge e_m \\ &= \det(A)\ e_1 \wedge \cdots \wedge e_m.\end{aligned}$$

□

**Remark 14.13.** In general, given an $m$-dimensional vector space $V$ and a linear map $L\colon V \to V$, the induced linear map

$$\Lambda^m L\colon \Lambda^m V \to \Lambda^m V$$

is multiplication by the base-independent quantity $\det(L)$. Fixing a basis for $V$ identifies $V$ with $\mathbb{F}^m$, and $L$ with a matrix multiplication $v \mapsto Av$. Then, the calculation above shows that $\Lambda^m L\colon \Lambda^m V \to \Lambda^m V$ is multiplication by $\det(A)$. Changing the basis replaces $A$ with a conjugate matrix $BAB^{-1}$, which has the same determinant.

**Exercise 14.10.** By the remark above, we need to fix a basis for $M_{2\times 2}(\mathbb{F})$, find the matrix of $L$ with respect to that basis, and calculate its determinant. Consider the standard basis

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

for $M_{2\times 2}(\mathbb{F})$. Then

$$\begin{aligned}
L(e_{11}) &= \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix} = b_{11}e_{11} + b_{21}e_{21}, \\
L(e_{12}) &= \begin{bmatrix} 0 & b_{11} \\ 0 & b_{21} \end{bmatrix} = b_{11}e_{12} + b_{21}e_{22}, \\
L(e_{21}) &= \begin{bmatrix} b_{12} & 0 \\ b_{22} & 0 \end{bmatrix} = b_{12}e_{11} + b_{22}e_{21}, \\
L(e_{22}) &= \begin{bmatrix} 0 & b_{12} \\ 0 & b_{22} \end{bmatrix} = b_{12}e_{12} + b_{22}e_{22},
\end{aligned}$$

which corresponds to the $4 \times 4$ matrix

$$\begin{bmatrix} b_{11} & 0 & b_{12} & 0 \\ 0 & b_{11} & 0 & b_{12} \\ b_{21} & 0 & b_{22} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{bmatrix}.$$

Switching two columns and two rows, we get the matrix

$$\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix},$$

which has the same determinant. We conclude that $\Lambda^4(L)$ is multiplication by

$$\det \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} = \det(B)^2.$$

$\square$

**Exercise 14.12.** If $\operatorname{rank} E = r$, then, by (14.2), we have $\operatorname{rank} \Lambda^k E = \binom{r}{k}$.

## Chapter 15

# Orientability

In the last lecture, we saw that given any real vector bundle $E \longrightarrow M$, the top exterior power defines a line bundle $\Lambda^{\text{top}} E \longrightarrow M$, whose elements correspond to wedges of vectors in a frame for $E$ at each point. The following definition will play an important role in defining the integral of differential forms over smooth manifolds later on.

**Definition 15.1.** The top exterior product line bundle $\Lambda^{\text{top}} E \longrightarrow M$ is also called the **determinant** line bundle of $E$ and is sometimes denoted by $\det(E)$. A real vector bundle is called **orientable** if and only if $\det(E)$ is isomorphic to the trivial line bundle $M \times \mathbb{R}$.

An **orientation** on a real vector bundle $E$ is a choice of isomorphism

$$\det(E) \longrightarrow M \times \mathbb{R}$$

up to multiplication by a positive function. A smooth manifold is called **orientable** if $TM$ is an orientable vector bundle; thus, an orientation on $M$ is a choice of isomorphism

$$\det(TM) \longrightarrow M \times \mathbb{R}$$

up to multiplication by a positive function.

**Remark 15.2.** By definition, for line bundles, being orientable is the same being trivial.

**Exercise 15.3.** Show that $\det(E \oplus E') \cong \det(E) \otimes \det(E')$.

**Exercise 15.4.** Use the previous exercise to prove the following: Suppose $M \subset N$ is a submanifold of codimension 1, and both $M$ and $N$ are orientable. Show that the normal bundle of $M$ in $N$ is trivial.

**Exercise 15.5.** Suppose that $M$ is an orientable smooth manifold and $E \longrightarrow M$ is an orientable smooth vector bundle. Use Exercises 13.10 and 15.3 to show that $E$ is an orientable manifold.

**Exercise 15.6.** Let $M$ and $N$ be nonempty smooth manifolds. Show that $M \times N$ is orientable if and only if both $M$ and $N$ are.

**Exercise 15.7.** Show that a vector bundle $E$ is orientable if and only if $E^*$ is orientable. Furthermore, a choice of orientation on $E$ determines an orientation on $E^*$.

The following proposition characterizes trivial line bundles, and thus orientable vector bundles.

**Proposition 15.8.** *Suppose $L \longrightarrow M$ is a real line bundle. Then $L$ is isomorphic to the trivial line bundle if and only if there exists a collection of local trivializations*

$$\Phi_\alpha \colon E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}$$

*over an open cover $\{U_\alpha\}$ of $M$ such that the corresponding change-of-trivialization maps*

$$\Phi_{\alpha\mapsto\beta} \colon U_\alpha \cap U_\beta \longrightarrow \mathbb{R}^*$$

*are positive. In particular, for every line bundle $L$, $L^{\otimes 2}$ is trivial.*

**Corollary 15.9.** *Suppose $E \longrightarrow M$ is a rank $r$ real vector bundle. Then $E$ is orientable if and only if there exists a collection of local trivializations*

$$\Phi_\alpha \colon E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^r$$

*over an open cover $\{U_\alpha\}$ of $M$ such that the corresponding change-of-trivialization maps*

$$\Phi_{\alpha\mapsto\beta} \colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(r, \mathbb{R})$$

*have positive determinant.*

**Proof.** The change-of-trivialization maps for $\det(E)$ are given by $\det(\Phi_{\alpha\mapsto\beta})$. If $E$ admits such a collection of local trivializations, then by Proposition 15.8, $\det(E)$ is isomorphic to the trivial line bundle.

Conversely, suppose $E$ is orientable, i.e., $\det(E)$ is isomorphic to the trivial line bundle, and fix an isomorphism

$$\varphi \colon \det(E) \longrightarrow M \times \mathbb{R}.$$

Starting from an arbitrary collection of local trivializations

$$\Phi_\alpha \colon E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^r$$

such that each $U_\alpha$ is connected, we compare the induced trivializations

$$\det(\Phi_\alpha) \colon \det(E)|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}$$

and the restrictions

$$\varphi|_{U_\alpha} \colon \det(E)|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}.$$

Since

$$(\varphi|_{U_\alpha}) \circ \det(\Phi_\alpha)^{-1} \colon U_\alpha \times \mathbb{R} \longrightarrow U_\alpha \times \mathbb{R}$$

is a bundle isomorphism, it is given by multiplication by a nowhere-vanishing function. Thus, over each $U_\alpha$, this function is either strictly positive or strictly negative. In the first case, we keep $\Phi_\alpha$ as is. In the second case, we replace $\Phi_\alpha$ with $\widetilde{\Phi}_\alpha = B \circ \Phi_\alpha$, where $B \in \mathrm{GL}(r, \mathbb{R})$ is any orientation-reversing linear isomorphism, such as

$$(x_1, \ldots, x_r) \mapsto (-x_1, x_2, \ldots, x_r).$$

This change flips the sign of the determinant, so that $(\varphi|_{U_\alpha}) \circ \det(\widetilde{\Phi}_\alpha)^{-1}$ becomes multiplication by a positive function. In this way, we obtain a new collection of local trivializations such that each

$$(\varphi|_{U_\alpha}) \circ \det(\Phi_\alpha)^{-1}$$

is multiplication by a positive function.

Since

$$(\varphi|_{U_\beta}^{-1} \circ \varphi|_{U_\alpha})|_{U_\alpha \cap U_\beta} = \mathrm{id},$$

it follows that

$$\begin{aligned}
\det(\Phi_{\alpha\mapsto\beta}) &= \det(\Phi_\beta \circ \Phi_\alpha^{-1}) \\
&= \det\left(\Phi_\beta \circ \varphi|_{U_\beta}^{-1} \circ \varphi|_{U_\alpha} \circ \Phi_\alpha^{-1}\right) \\
&= \det\left((\varphi|_{U_\beta} \circ \det(\Phi_\beta)^{-1})^{-1}\right) \det\left((\varphi|_{U_\alpha} \circ \det(\Phi_\alpha)^{-1})\right) > 0,
\end{aligned}$$

as desired. □

**Proof of Proposition 15.8.** While one can prove this proposition more easily by introducing a Riemannian metric on $L$, we give here a self-contained argument that avoids the use of a metric. Instead, we rely directly on a partition of unity – a technique that will later reappear in the construction of Riemannian metrics.

One direction is straightforward. If $\psi \colon L \longrightarrow M \times \mathbb{R}$ is an isomorphism, then it defines a global trivialization of $L$, and its restriction to any open set of $M$ gives a local trivialization for which all transition functions are the identity map (and hence positive).

Conversely, suppose we are given a collection of local trivializations

$$\Phi_\alpha \colon L|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}$$

over an open cover $\{U_\alpha\}$ of $M$, such that the associated change-of-trivialization maps

$$\Phi_{\alpha\mapsto\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathbb{R}^*$$

are all positive-valued functions. An isomorphism $\psi\colon L \longrightarrow M \times \mathbb{R}$ arises from a collection of local isomorphisms

$$U_\alpha \times \mathbb{R} \longrightarrow U_\alpha \times \mathbb{R}, \qquad (x, v) \mapsto (x, f_\alpha(x)v),$$

where the functions $f_\alpha\colon U_\alpha \to \mathbb{R}_+$ are required to satisfy the compatibility condition

$$f_\beta \cdot \Phi_{\alpha\mapsto\beta} = f_\alpha \quad \text{on } U_\alpha \cap U_\beta.$$

In other words, on overlaps $U_\alpha \cap U_\beta$, the following diagram commutes:

$$\begin{tikzcd}
L|_{U_\alpha} \arrow[r, "\Phi_\alpha"] & U_\alpha \times \mathbb{R} \arrow[r, "\times f_\alpha"] \arrow[d, "\times \Phi_{\alpha\mapsto\beta}"] & U_\alpha \times \mathbb{R} \arrow[d, "\mathrm{id}"] \\
L|_{U_\beta} \arrow[r, "\Phi_\beta"] & U_\beta \times \mathbb{R} \arrow[r, "\times f_\beta"] & U_\beta \times \mathbb{R}
\end{tikzcd}$$

Since each $\Phi_{\alpha\mapsto\beta}$ is a positive function, we can define

$$h_{\alpha\mapsto\beta} := \log(\Phi_{\alpha\mapsto\beta}),$$

and seek functions $h_\alpha = \log(f_\alpha)\colon U_\alpha \to \mathbb{R}$ such that the equivalent compatibility condition

$$h_\alpha = h_{\alpha\mapsto\beta} + h_\beta \quad \text{on } U_\alpha \cap U_\beta$$

is satisfied. Let $\{\varrho_\alpha\colon U_\alpha \to [0,1]\}$ be a partition of unity subordinate to the given cover. Then the functions

$$h_\alpha = \sum_{\beta\neq\alpha} \varrho_\beta \cdot h_{\alpha\mapsto\beta}$$

are well-defined and smooth, and they satisfy the required relation on overlaps. Thus, the functions $f_\alpha = e^{h_\alpha}$ define a global trivialization of $L$.

Given a collection of local trivializations

$$\Phi_\alpha\colon L|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}$$

of $L$ over an open cover $\{U_\alpha\}$ of $M$, the induced local trivializations of $L^{\otimes 2}$ are given by $\Phi_\alpha^{\otimes 2}$. Consequently, the change-of-trivialization maps for $L^{\otimes 2}$ are the squares of those for $L$, and are therefore positive-valued.

□

**Remark 15.10.** The proof of the following statement is somewhat beyond the scope of this course, but real line bundles over a manifold $M$ are classified topologically by a homological invariant called the **first Stiefel–Whitney class**.

If $\widetilde{M}$ is the universal cover of $M$, then every real line bundle over $M$ is the quotient of the trivial line bundle $\widetilde{M} \times \mathbb{R}$, where the action of $\pi_1(M)$ on the second factor is determined by a group homomorphism

$$\pi_1(M) \longrightarrow \mathbb{R}^* = \mathrm{Auto}(\mathbb{R}).$$

Since $\mathbb{R}^*$ is abelian, this homomorphism factors through the abelianization of $\pi_1(M)$, giving a map

$$H_1(M; \mathbb{Z}) \longrightarrow \mathbb{R}^*.$$

Here, $H_1(M; \mathbb{Z})$ is the first homology group of $M$ that we do not define in this book. Composing with the group homomorphism $\mathbb{R}^* \to \mathbb{Z}_2$ that sends $t \neq 0$ to the sign of $t$, we obtain a homomorphism

$$H_1(M; \mathbb{Z}) \longrightarrow \mathbb{Z}_2.$$

Two elements $\varphi, \varphi' \in \mathrm{Hom}(H_1(M), \mathbb{R}^*)$ define isomorphic real line bundles if and only if they induce the same homomorphism $H_1(M) \to \mathbb{Z}_2$. (See Exercise 15.7 for a related discussion.)

Moreover, the real line bundle associated to $\varphi \in \mathrm{Hom}(H_1(M), \mathbb{Z}_2)$ is topologically trivial if and only if $\varphi$ is the trivial homomorphism.

**Example 15.11.** For $M = S^1$, we have $\pi_1(S^1) = \mathbb{Z}$ and $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z}_2$. The non-trivial line bundle $L \to S^1 = \mathbb{R}/\mathbb{Z}$ associated to the unique non-trivial homomorphism is the Möbius band:

$$L = (\mathbb{R} \times \mathbb{R})/\mathbb{Z},$$

where $n \in \mathbb{Z}$ acts by $(x, v) \sim (x + n, (-1)^n v)$ for all $(x, v) \in \mathbb{R} \times \mathbb{R}$.

**Exercise 15.12.** If $E \longrightarrow M$ is a vector bundle, show that $E \oplus E$ is orientable.

**Exercise 15.13.** Show that every complex vector bundle, when seen as a real vector bundle, is orientable.

**Exercise 15.14.** Show that the tangent bundle of any smooth manifold is an orientable manifold.

**Exercise 15.15.** By studying the transition maps of the standard atlas of $\mathbb{RP}^2$, prove that $\mathbb{RP}^2$ is not orientable (i.e. prove $\mathbb{RP}^2$ is not orientable without using $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$).

**Definition 15.16.** Suppose $E \longrightarrow M$ is an orientable vector bundle and fix an orientation on it; that is, fix a line bundle isomorphism $\psi\colon \Lambda^{\mathrm{top}} E \longrightarrow$

$M \times \mathbb{R}$ up to multiplication by a positive function. For every subset $U \subset M$ and any frame $s_1, \ldots, s_k$ for $E|_U$, we say $(s_1, \ldots, s_k)$ is a **positive** frame if

$$\psi_x(s_1(x) \wedge \cdots \wedge s_k(x)) \in \mathbb{R}_+ \qquad \forall\, x \in U;$$

i.e., $\psi$ maps $s_1 \wedge \cdots \wedge s_k$ to a positive multiple of the constant section 1 in $U \times \mathbb{R}$.

**Definition 15.17.** Suppose $M$ is an orientable manifold and fix an orientation on $M$; that is, fix an isomorphism $\psi\colon \Lambda^{\mathrm{top}}TM \longrightarrow M \times \mathbb{R}$ up to multiplication by a positive function. We say a chart $\varphi\colon U \longrightarrow V \subset \mathbb{R}^m$ is **compatible with the orientation** if $(\partial_{x_1}, \ldots, \partial_{x_m})$ is a positive frame. An atlas on $M$ is called an **oriented atlas** if every chart in it is compatible with the orientation.

If a chart is not compatible with the orientation, we can compose it with an orientation-reversing diffeomorphism such as

$$(x_1, \ldots, x_m) \mapsto (-x_1, \ldots, x_m)$$

to make it compatible. Two overlapping charts $\varphi_\alpha\colon U_\alpha \longrightarrow \mathbb{R}^m$ and $\varphi_\beta\colon U_\beta \longrightarrow \mathbb{R}^m$ are both compatible with the orientation or not if and only if

$$\det(d\varphi_{\alpha\mapsto\beta}) > 0.$$

On every oriented smooth manifold with the maximal smooth atlas $\mathcal{A}$, we can choose a maximal oriented subatlas $\mathcal{A}^+$ of charts compatible with the orientation. By the discussion above, the remaining charts form a subatlas $\mathcal{A}^-$ that is the maximal oriented subatlas for the **opposite orientation**. The latter corresponds to the trivialization

$$-\psi\colon \Lambda^{\mathrm{top}}TM \longrightarrow M \times \mathbb{R}.$$

**Exercise 15.18.** Is the two-chart atlas (2.3) on $S^2$ an oriented atlas?

**Exercise 15.19.** Prove that if $M$ is an orientable smooth manifold (with boundary) then the boundary $\partial M$ is also an orientable manifold. Describe a convention for defining an induced orientation on $\partial M$.

Orientation will play a major role in the definition of the integral of differential forms on manifolds, and we will use an oriented atlas for related calculations.

## Solutions to exercises

**Exercise 15.3.** Suppose $\Phi_{\alpha\mapsto\beta}$ are the change-of-trivialization matrix-valued functions of $E$ and $\Phi'_{\alpha\mapsto\beta}$ are the change-of-trivialization matrix-valued functions of $E'$ with respect to a collection of local trivializations over the same open covering of $M$. Then, recall from (13.1) that the change-of-trivialization matrix-valued functions corresponding to $E \oplus E'$ are the block-diagonal matrices

$$\begin{bmatrix} \Phi_{\alpha\mapsto\beta} & 0 \\ 0 & \Phi'_{\alpha\mapsto\beta} \end{bmatrix}.$$

Then, as we explained in the proof of Lemma 14.11, the change-of-trivialization functions of the line bundles $\det(E)$, $\det(E')$, and $\det(E \oplus E')$ are

$$\det(\Phi_{\alpha\mapsto\beta}), \quad \det(\Phi'_{\alpha\mapsto\beta}), \quad \text{and} \quad \det\begin{bmatrix} \Phi_{\alpha\mapsto\beta} & 0 \\ 0 & \Phi'_{\alpha\mapsto\beta} \end{bmatrix} = \det(\Phi_{\alpha\mapsto\beta})\det(\Phi'_{\alpha\mapsto\beta}),$$

respectively. Since the latter is the product of the first two, we conclude that

$$\det(E \oplus E') \cong \det(E) \otimes \det(E').$$

□

**Exercise 15.4.** We know from Exercise 13.8 that $TN|_M$ splits as

$$TN|_M \cong TM \oplus \mathcal{N}_N M.$$

By the previous exercise, and since $\mathcal{N}_N M$ is a line bundle (so $\Lambda^{\text{top}}(\mathcal{N}_N M) = \Lambda^1(\mathcal{N}_N M) = \mathcal{N}_N M$), we have

$$\Lambda^{\text{top}}(TN|_M) \cong \Lambda^{\text{top}}(TM) \otimes \mathcal{N}_N M.$$

By the orientability assumption, both $\Lambda^{\text{top}}(TN)$ and $\Lambda^{\text{top}}(TM)$ are trivial line bundles. Therefore, $\mathcal{N}_N M$ is also trivial. □

**Exercise 15.5.** By Exercise 13.10, we have

$$TE \cong \pi^*(E \oplus TM).$$

By Exercise 15.3 and since taking the exterior product commutes with pull-back, we get

$$\Lambda^{\text{top}} TE = \pi^* \Lambda^{\text{top}}(E \oplus TM) = \pi^*\big(\Lambda^{\text{top}} E \otimes \Lambda^{\text{top}} TM\big).$$

By assumption, both $E$ and $M$ are orientable; i.e., $\Lambda^{\text{top}} E$ and $\Lambda^{\text{top}} TM$ are trivial line bundles. We conclude that $\Lambda^{\text{top}} TE$ is also trivial; i.e., $E$ is an orientable manifold. □

**Exercise 15.6.** If $M$ and $N$ are orientable, then $\det(TM)$ and $\det(TN)$ are trivial line bundles. Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ denote the projection maps to the first and second factors. Then,

$$T(M \times N) = \pi_M^* TM \oplus \pi_N^* TN.$$

Therefore,

$$\det(T(M \times N)) = \pi_M^* \det(TM) \otimes \pi_N^* \det(TN)$$

is also trivial. We conclude that $M \times N$ is orientable.

Conversely, suppose $M \times N$ is orientable. For any $y \in N$, restricted to $M \times \{y\} \cong M$, we have

$$T(M \times N)|_{\{x\} \times N} \cong TM \oplus M \times T_y N.$$

The second term on the right is a trivial (product) vector bundle; therefore, $\Lambda^{\text{top}}(M \times T_y N)$ is the trivial line bundle. Since

$$\det\left(T(M \times N)|_{\{x\} \times N}\right) \cong \det(TM) \otimes \det(M \times T_y N),$$

and both the left-hand term and the last term are trivial, we conclude that $\Lambda^{\text{top}} TM$ is also trivial. Therefore, $M$ is orientable. By symmetry of the argument, $N$ is orientable as well. □

**Exercise 15.7.** Since $\det(E^*) = \det(E)^*$, taking duals and then inverting, a choice of isomorphism $\det(E) \to \mathcal{O}$ determines an isomorphism $\det(E^*) \to \mathcal{O}$. □

**Exercise 15.12.** We have

$$\det(E \oplus E) = \det(E) \otimes \det(E) = \det(E)^{\otimes 2}.$$

By Proposition 15.8, $\det(E)^{\otimes 2} \cong M \times \mathbb{R}$. We conclude that $E \oplus E$ is orientable. □

**Exercise 15.13.** Given a collection of local trivializations

$$\Phi_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C}^r$$

over an open cover $\{U_\alpha\}$ of $M$, let $\Phi_{\alpha \mapsto \beta} \in \mathrm{GL}(r, \mathbb{C})$ denote the change-of-trivialization matrix-valued functions of $E$. Let

$$\Phi_{\alpha \mapsto \beta} = \Phi'_{\alpha \mapsto \beta} + \mathfrak{i}\Phi''_{\alpha \mapsto \beta}$$

denote the decomposition into real and imaginary parts. Then the change-of-trivialization matrix-valued functions of the real vector bundle underlying

$E$ are

$$\Phi^{\mathbb{R}}_{\alpha\mapsto\beta} = \begin{bmatrix} \Phi'_{\alpha\mapsto\beta} & -\Phi''_{\alpha\mapsto\beta} \\ \Phi''_{\alpha\mapsto\beta} & \Phi'_{\alpha\mapsto\beta} \end{bmatrix}.$$

It is an exercise in linear algebra that

$$\det(\Phi^{\mathbb{R}}_{\alpha\mapsto\beta}) = \det(\Phi'_{\alpha\mapsto\beta})^2 + \det(\Phi''_{\alpha\mapsto\beta})^2 > 0.$$

Therefore, by Corollary 15.9, the real vector bundle underlying $E$ is orientable. □

**Exercise 15.14.** Applying Exercise 13.10 to $E = TM$, we have

$$T(TM) = \pi^*(TM \oplus TM).$$

Therefore,

$$\det(T(TM)) = \pi^* \det(TM)^{\otimes 2}.$$

By Proposition 15.8, $\det(T(TM)) \cong TM \times \mathbb{R}$. We conclude that $TM$ is an orientable manifold. □

**Exercise 15.15.** Recall from Section 2 and the solution to Exercise 3.5 that $\mathbb{RP}^2$ can be covered by three charts $\varphi_j \colon U_j \to V_j \cong \mathbb{R}^n$ (respectively, $\mathbb{C}^n$), for $j = 0, 1, 2$, with transition maps given by

$$\varphi_{i\mapsto j} = \varphi_j \circ \varphi_i^{-1}\big((x_k)_{k\neq i}\big) = (y_k)_{k\neq j}, \quad \text{where} \quad y_k = \begin{cases} x_k/x_j & \text{if } k \neq i, \\ 1/x_j & \text{if } k = i. \end{cases}$$

In particular, if $(x_1, x_2)$ denote the coordinates on $V_0$ and $(y_0, y_1)$ denote the coordinates on $V_2$, then

$$(y_0, y_1) = \varphi_{0\mapsto 2}(x_1, x_2) = (1/x_2, x_1/x_2).$$

Therefore,

$$d\varphi_{0\mapsto 2} = \begin{bmatrix} 0 & -1/x_2^2 \\ 1/x_2 & -x_1/x_2^2 \end{bmatrix}.$$

We conclude that

$$\det(d\varphi_{0\mapsto 2}) = x_2^{-3}.$$

The overlap region $V_{0,2} = \mathbb{R} \times \mathbb{R}^*$ has two connected components, on one of which $x_2^{-3}$ is positive and on the other it is negative. Therefore, since $V_0$ and $V_2$ are connected, there are no local trivializations

$$\psi_0 \colon \Lambda^2(TV_0) \longrightarrow V_0 \times \mathbb{R}, \quad \text{and} \quad \psi_2 \colon \Lambda^2(TV_2) \longrightarrow V_2 \times \mathbb{R}$$

such that

$$\psi_2 \circ \Lambda^2(d\varphi_{0\mapsto 2}) \circ \psi_0^{-1} \colon V_{0,2} \times \mathbb{R} \longrightarrow V_{2,0} \times \mathbb{R}$$

is positive. □

**Exercise 15.18.** As shown in Exercise 3.5, the transition map is

$$\varphi_{+\mapsto-}(x) = \frac{1}{|x|^2}x.$$

Withing $r = |x|$, we have

$$d\varphi_{+\mapsto-} = \frac{1}{r^4}\begin{bmatrix} r^2 - 2x_1^2 & -2x_1x_2 & \cdots & -2x_1x_m \\ -2x_1x_2 & r^2 - 2x_2^2 & \cdots & -2x_2x_m \\ \vdots & & \ddots & \vdots \\ -2x_1x_m & \cdots & & r^2 - 2x_m^2 \end{bmatrix} = \frac{1}{r^2}\mathrm{I}_m - \frac{2}{r^4}xx^T$$

where $x$ is treated as a column vector. Since $x \neq 0$, the $m \times m$ matrix $xx^T$ has rank 1. Moreover, $\frac{1}{r^2}xx^T$ is the orthogonal projection matrix onto the span of $x$. Therefore,

$$\mathrm{I}_m - \frac{2}{r^2}xx^T$$

is the matrix of reflection with respect to the plane $x^\perp$ and has determinant $-1$. We conclude that

$$\det(d\varphi_{+\mapsto-}) = -r^{-2m} < 0.$$

Therefore, the two-chart atlas (2.3) on $S^2$ is not an oriented atlas. □

**Exercise 15.19.** The boundary $\partial M$ of $M$ is a manifold of real codimension one, and we have a short exact sequence of vector bundles

$$0 \longrightarrow T\partial M \longrightarrow TM|_{\partial M} \longrightarrow \mathcal{N}_M\partial M \longrightarrow 0,$$

where

$$\mathcal{N}_M\partial M$$

is the normal bundle of $\partial M$ in $M$. For every $p \in \partial M$, there are two distinguished directions in $\mathcal{N}_M\partial M|_p \cong \mathbb{R}$: one pointing toward the interior of $M$, and the opposite, outward direction.

More precisely, if $\varphi\colon U \to V \subset \mathbb{H}_m$ is a chart around $p$ such that $\varphi(\partial M \cap U) \subset \partial\mathbb{H}_m$ in the sense of Definition 1.6, then the inward direction corresponds to (the image of) any vector field of the form $u + f\partial_{x_1}$ along $\partial\mathbb{H}_m$ (in the quotient bundle $\mathcal{N}_M\partial M$), where $u$ is tangent to $\partial\mathbb{H}_m$ and $f > 0$. Similarly, the outward direction corresponds to such a vector field with $f < 0$.

Therefore, $\mathcal{N}_M\partial M$ admits a collection of local trivializations

$$\Phi\colon \mathcal{N}_M\partial M|_{U\cap\partial M} \longrightarrow (U \cap \partial M) \times \mathbb{R}$$

such that $\Phi^{-1}(e_1)$ corresponds to the outward direction, where $e_1$ denotes the constant section 1 of the trivial bundle. For these trivializations, the transition maps are positive. Thus, by Proposition 15.8, the line bundle $\mathcal{N}_M\partial M$ is trivial.

Moreover, there is a natural correspondence between the two possible trivializations of $\mathcal{N}_M \partial M$ (on each connected component of $\partial M$) and the choice of outward or inward vector fields along $\partial M$.

Suppose $\det(TM) \to M \times \mathbb{R}$ is a trivialization that describes the chosen orientation on $TM$. Along $\partial M$, the short exact sequence above induces a canonical isomorphism

$$\det(TM)|_{\partial M} \cong \det(T\partial M) \otimes \mathcal{N}_M \partial M.$$

Since $M$ is orientable, $\det(TM)$ is isomorphic to the trivial line bundle. From the discussion above, $\mathcal{N}_M \partial M$ is also trivial. We conclude that $\det(T\partial M)$ is trivial as well, and therefore $\partial M$ is orientable.

To describe a convention for defining the induced orientation on $\partial M$, we need to specify when a frame for $T\partial M$ (at a point on the boundary) is considered oriented. Given an orientation on $TM$, we define the induced orientation on $T\partial M$ so that the isomorphism

$$T_p M \cong \mathbb{R} \cdot \vec{n}_{\text{out}}(p) \oplus T_p \partial M, \qquad \forall p \in \partial M,$$

is consistent with the orientations on both sides. Here, $\vec{n}_{\text{out}}(p)$ is any outward-pointing vector in $T_p M$. In other words, $(v_1, \ldots, v_{m-1}) \in T_p \partial M$ is a positively oriented frame for $T_p \partial M$ if and only if

$$(n_{\text{out}}(p), v_1, \ldots, v_{m-1}) \in T_p M$$

is a positively oriented frame for $T_p M$. $\square$

Chapter 16

# Metric

We start by recalling the following definition briefly discussed in Example 14.7.

Given a continuous or smooth real vector bundle $E \longrightarrow M$, let $E \times_M E$ denote the fiber product of $E$ with itself with respect to the projection map $\pi\colon E \longrightarrow M$ in the sense of Theorem 9.4. In other words, the fiber of $E \times_M E$ over $p \in M$ is $E_p \times E_p$.

**Definition 16.1.** Given a continuous or smooth real vector bundle $E \longrightarrow M$, a **Riemannian metric** $g$ on $E$ is a symmetric fiber-wise bilinear map

$$g\colon E \times_M E \longrightarrow \mathbb{R}$$

that is positive-definite in the sense that

$$g(v,v) > 0 \qquad \forall\, p \in M,\ 0 \neq v \in E_p.$$

A Riemannian metric on a smooth manifold $M$ is a Riemannian metric on its tangent bundle.

A Riemannian metric on a smooth vector bundle provides a smoothly varying inner product on each fiber, enabling us to carry out geometry in a precise and intrinsic way. It allows us to define notions of length, angle, and orthogonality for sections of the bundle, which are essential for both geometric and analytic constructions. In particular, a Riemannian metric lets us decompose each fiber into orthogonal subspaces – for example, identifying the quotient $E/E'$ of a vector bundle embedding $E' \subset E$ with the orthogonal complement of $E'$ in $E$. This yields a simpler and more intuitive proof that every short exact sequence of smooth vector bundles splits.

Given a local trivialization

$$\Phi\colon E|_U \longrightarrow U \times \mathbb{R}^k,$$

the metric takes the matrix form

$$(u,v) \mapsto u^T \mathcal{G}(x)\, v \qquad \forall\; u,v \in \{x\} \times \mathbb{R}^k,$$

where each $\mathcal{G}(x)$ is a continuous or smooth family of $k \times k$ symmetric positive-definite matrices (depending on $\Phi$) and $u, v$ are column vectors.

Given a collection of local trivializations

$$\Phi_\alpha\colon E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^k,$$

over an open cover $\{U_\alpha\}$ of $M$, let $\Phi_{\alpha\mapsto\beta} \in \mathrm{GL}(k,\mathbb{R})$ denote the change-of-trivialization matrix-valued functions of $E$, and let $\mathcal{G}_\alpha$ denote the matrix form of $g$ with respect to $\Phi_\alpha$. Then,

$$\mathcal{G}_\alpha = \Phi^T_{\alpha\mapsto\beta}\, \mathcal{G}_\beta\, \Phi_{\alpha\mapsto\beta} \qquad \forall\; \alpha, \beta. \tag{16.1}$$

Conversely, a collection $\{\mathcal{G}_\alpha\}$ of positive-definite $k \times k$ matrix-valued functions on $\{U_\alpha\}$ satisfying the compatibility relation above defines a well-defined metric $g$ on $E$.

**Lemma 16.2.** *Every vector bundle admits (a plethora of) Riemannian metrics.*

**Proof.** Consider an arbitrary collection of local trivializations

$$\Phi_\alpha\colon E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^k,$$

over an open cover $\{U_\alpha\}$ of $M$, and equip each $E|_{U_\alpha}$ with the metric $g_\alpha$ corresponding to the standard Riemannian metric on $U_\alpha \times \mathbb{R}^k$ (i.e., $\mathcal{G}_\alpha = \mathrm{I}_k$). Let $\{\varrho_\alpha\colon U_\alpha \longrightarrow [0,1]\}$ be a partition of unity subordinate to the open covering in consideration. Then the expression

$$g = \sum_\alpha \varrho_\alpha g_\alpha$$

is well-defined and defines a Riemannian metric on $E$. □

**Exercise 16.3.** Use a Riemannian metric on any line bundle $L \to M$, to prove that $L$ admits a collection of local trivializations $L|_{U_\alpha} \cong U_\alpha \times \mathbb{R}$ over an open cover $\{U_\alpha\}$ of $M$ such that the change of trivialization maps

$$\begin{aligned} &\Phi_{\alpha\mapsto\beta}\colon U_\alpha \cap U_\beta \to \mathbb{R}^*,\\ &(U_\alpha \cap U_\beta) \times \mathbb{R} \ni (x,v) \to (x, \Phi_{\alpha\mapsto\beta}(x) v) \in (U_\alpha \cap U_\beta) \times \mathbb{R} \end{aligned}$$

are constant $\pm 1$.

**Exercise 16.4.** Generalize the previous result to show that on every vector bundle $E \to M$, there exists a collection of local trivializations $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ over an open cover $\{U_\alpha\}$ of $M$ such that the matrix-valued change of trivialization maps

$$\Phi_{\alpha\mapsto\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k,\mathbb{R}),$$
$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \ni (x,v) \to (x, \Phi_{\alpha\mapsto\beta}(x)v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

take values in $\mathrm{O}(k) \subset \mathrm{GL}(k,\mathbb{R})$. Further, if the vector bundle is orientable, we can improve that to $\mathrm{SO}(k)$.

**Remark 16.5.** Note that $\mathrm{O}(1) = \{\pm 1\}$ and $\mathrm{SO}(1) = \{+1\}$. Therefore, the statement above implies Proposition 15.8.

**Exercise 16.6.** Show that a Riemannian metric on a vector bundle $E$ induces a Riemannian metric on its dual $E^*$, and more generally on all tensor/exterior products of $E$.

**Definition 16.7.** Suppose $E \to M$ is a rank $r$ real vector bundle equipped with a Riemannian metric $g$, and $s_1, \ldots, s_r$ is a frame for $E|_U$. We say $s_1, \ldots, s_r$ is an **orthonormal frame** if $|s_i(p)| = 1$ for all $i = 1, \ldots, r$ and $p \in M$, and

$$g(s_i(p), s_j(p)) = 0 \quad \text{for all } i \neq j \text{ and } p \in M.$$

Here,

$$|v| = \sqrt{g(v,v)} \qquad \forall\, p \in M,\ v \in E_p$$

is the length of a vector with respect to $g$. Also, the angle between two non-zero vectors $v, v' \in E_p$ is given by

$$\cos^{-1}\left(\frac{g(v,v')}{|v||v'|}\right) \in [0,\pi],$$

and the condition $g(s_i(p), s_j(p)) = 0$ means that $s_i(p)$ and $s_j(p)$ are perpendicular.

**Exercise 16.8.** Suppose $s_1, \ldots, s_r$ and $s'_1, \ldots, s'_r$ are two (local) orthonormal frames for $(E, g)$. Show that

$$s_1 \wedge \ldots \wedge s_r = \pm s'_1 \wedge \ldots \wedge s'_r \in \Lambda^{\mathrm{top}} E.$$

Moreover, if $E$ is oriented and both are positive frames in the sense of Definition 15.16, then

$$s_1 \wedge \ldots \wedge s_r = s'_1 \wedge \ldots \wedge s'_r \in \Lambda^{\mathrm{top}} E.$$

**Remark 16.9.** The previous exercises says that if $E$ is oriented, then every Riemannian metric $g$ on $E$ determines a unique global section of $\det(E)$ (and thus a unique trivialization of $\det(E)$) that has norm one with respect to the induced metric on $\det(E)$.

Recall that a metric on a manifold $M$ is a metric $g$ on its tangent bundle. Every chart $\varphi\colon U \longrightarrow V \subset \mathbb{R}^m$ on $M$ determines a local trivialization of the tangent bundle with respect to which the metric can be expressed in the matrix form

$$g(u,v) = u^T \mathcal{G}\, v, \qquad \mathcal{G} = \Big[ g_{ij} = g(\partial_{x_i}, \partial_{x_j}) \Big],$$

such that $\big[g_{ij}\big]$ is a symmetric positive definite matrix depending on the variable $x \in V \subset \mathbb{R}^m$. As a symmetric tensor $g \in \Gamma(M, \mathrm{Sym}^2(T^*M))$, $g$ has the local equation

$$g|_U = \sum_{i,j} g_{ij}(x)\ dx_i \otimes dx_j,$$

for which we are viewing $g$ as a symmetric tensor in the sense of Definition 14.6.

**Exercise 16.10.** If $M \subset N$ is a submanifold, every metric on $TN$ induces a metric on $TM \subset TN|_M$ (by restriction). Consider the two-chart covering of $S^2$ and find the $2 \times 2$ matrices of the metric induced by the standard metric on $\mathbb{R}^3$ to $S^2$ in each chart.

A Riemannian metric $g$ on a vector bundle $E$ yields an isomorphism between $E$ and $E^*$ by mapping

$$v \in E_p \longmapsto g(v, \cdot) \in E_p^* = \mathrm{Hom}(E_p, \mathbb{R}).$$

In the case of a metric on a manifold $M$, this identifies $TM$ with $T^*M$ and therefore their sections as well; i.e., vector fields with differential 1-forms. For instance, recall from Section 12 that associated to every smooth function $f\colon M \to \mathbb{R}$, we have the differential 1-form $df$, which locally expands as

$$df = \sum_i \frac{\partial f}{\partial x_i}\ dx_i.$$

By the identification above, the vector field $\nabla f$ associated to $f$, called the **gradient vector field**, satisfies

$$g(\nabla f, \cdot) = df. \tag{16.2}$$

Suppose $\nabla f$ has the local expansion

$$\nabla f = \sum_i a_i\ \partial_{x_i}.$$

To determine the coefficients $a_i$, apply both sides to the basis vectors $\partial_{x_j}$. We obtain

$$g(\nabla f, \partial_{x_j}) = \sum_{i=1}^m a_i\, g(\partial_{x_i}, \partial_{x_j}) = \sum_{i=1}^m g_{ij} a_i = df(\partial_{x_j}) = \frac{\partial f}{\partial x_j}.$$

Since the matrix $G = [g_{ij}]$ is symmetric, the above equations for all $j = 1, \ldots, m$ are encoded in the matrix equation

$$\mathcal{G}\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{bmatrix}.$$

We conclude that

$$\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \mathcal{G}^{-1}\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{bmatrix}.$$

It is customary to denote the inverse of $\mathcal{G}$ by $\left[g^{ij}\right]$. Therefore, the gradient vector field of any smooth function $f$ has the local expression

$$\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_i}\, \partial_{x_j}$$

in any coordinate chart.

If $Y \subset M$ is a level set of the smooth function $f$, then $T_yY = \ker(df|_y)$ coincides with the orthogonal complement of $\nabla f(y)$, because

$$df(v) = 0 \quad \Longleftrightarrow \quad g(\nabla f, v) = 0.$$

We will talk more about the correspondence between vector fields and differential forms in the future lectures.

**Exercise 16.11.** The length of a parametrized path

$$\gamma\colon (a, b) \to M$$

into a Riemannian manifold $(M, g)$ is the quantity

$$|\gamma| = \int_a^b |\dot\gamma(t)|\, dt,$$

where $\dot\gamma(t) = \frac{d\gamma}{dt} \in T_{\gamma(t)}M$ is the velocity vector.

Calculate the length of a semicircle of radius $r$ centered at the origin in the upper half-plane

$$\mathcal{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with respect to the metric

$$g = \frac{dx \otimes dx + dy \otimes dy}{y^2}.$$

Continuous, smooth, or holomorphic vector bundles can similarly be equipped with a Hermitian metric, which generalizes the standard Hermitian inner product on $\mathbb{C}^n$, given by

$$\mathbb{C}^n \ni u, v \longmapsto u^T \overline{v} \in \mathbb{C},$$

to arbitrary complex vector bundles. Due to the presence of complex conjugation in the definition, every Hermitian metric is a smooth object, even when defined on a holomorphic vector bundle, and the notion of a holomorphic metric is thus meaningless.

**Remark 16.12.** Some sources define the standard Hermitian inner product on $\mathbb{C}^n$ to be

$$\mathbb{C}^n \ni u, v \longmapsto \overline{u}^T v \in \mathbb{C}.$$

Adopting this convention will affect some parts of the definition below.

**Definition 16.13.** Given a continuous, smooth, or holomorphic complex vector bundle $E \to M$, a **Hermitian metric** $\mathfrak{h}$ on $E$ is a fiberwise map

$$\mathfrak{h} \colon E \times_M E \to \mathbb{C}$$

that is complex linear in the first factor, anti-complex linear in the second factor, positive-definite in the sense that

$$\mathfrak{h}(v, v) > 0 \qquad \forall\, p \in M,\ 0 \neq v \in E_p,$$

and conjugate-symmetric in the sense that

$$\mathfrak{h}(u, v) = \overline{\mathfrak{h}(v, u)} \qquad \forall\, p \in M,\ u, v \in E_p.$$

A Hermitian metric on a smooth manifold $M$ with a complex tangent bundle is a Hermitian metric on $TM$.

**Remark 16.14.** The real part $g = \mathfrak{h}^{\mathbb{R}}$ of any Hermitian metric $\mathfrak{h}$ is a Riemannian metric on the underlying real vector space of $E$. Note that $\mathfrak{h}(v, v) \in \mathbb{R}$ for all $v \in E$, so the length of a vector can be computed using either $\mathfrak{h}$ or $\mathfrak{h}^{\mathbb{R}}$. The concept of angle, however, is defined using $\mathfrak{h}^{\mathbb{R}}$ alone.

**Example 16.15.** Suppose $M$ is a holomorphic manifold and $\mathfrak{h}$ is a Hermitian metric on its complex tangent bundle $TM$. With respect to any holomorphic chart $\varphi \colon U \longrightarrow V \subset \mathbb{C}^m$ on $M$ the metric $\mathfrak{h}$ takes the form

$$\mathfrak{h}|_U = \sum_{i,j} \mathfrak{h}_{ij}(z)\ dz_i \otimes \overline{dz_j}$$

where $H \coloneqq [\mathfrak{h}_{ij}]$ is a positive definite Hermitian matrix (i.e. $\overline{H}^T = H$) depending on the complex variables $z = (z_1, \dots, z_m) \in V \subset \mathbb{R}^m$.

**Exercise 16.16.** Show that the real part of the Hermitian metric

$$\mathfrak{h} = \frac{dz \otimes \overline{dz}}{\mathrm{Im}(z)^2}$$

on the upper half plane $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ coincides with the Riemannian metric in Exercise 16.11. Show that the action of $\mathrm{SL}(2, \mathbb{R})$ in Exercise 5.6 on $\mathcal{H}$ is an **isometry**, meaning that the metric $\mathfrak{h}$ is preserved under the action of elements of $\mathrm{SL}(2, \mathbb{R})$.

With a proof identical to that of Lemma 16.2, one can show that every smooth complex vector bundles admits a plethora of smooth Hermitian metrics.

**Exercise 16.17.** Show that every continuous or smooth complex vector bundle $E \to M$ admits a collection of local trivializations $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^k$ such that the matrix-valued transition maps

$$\Phi_{\alpha\mapsto\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C}),$$
$$(U_\alpha \cap U_\beta) \times \mathbb{C}^k \ni (x, v) \to (x, \Phi_{\alpha\mapsto\beta}(x)v) \in (U_\alpha \cap U_\beta) \times \mathbb{C}^k,$$

take values in $\mathrm{U}(k)$.

## Solutions to exercises

**Exercise 16.3.** Fix a metric $g$ on $L$. Start with an arbitrary collection of local trivializations $\Phi_\alpha\colon L|_{U_\alpha} \cong U_\alpha \times \mathbb{R}$ over an open cover $\{U_\alpha\}$ of $M$ and let

$$f_\alpha(x) = g(\Phi_\alpha^{-1}(x,1), \Phi_\alpha^{-1}(x,1)) \in \mathbb{R}_+ \qquad \forall x \in U_\alpha.$$

Multiplying $\Phi_\alpha$ with $\sqrt{f_\alpha}$ defines a new collection of trivializations

$$\widetilde{\Phi}_\alpha = \sqrt{f_\alpha} \cdot \Phi_\alpha\colon L|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}$$

that identifies $g|_{U_\alpha}$ with the standard metric on $U_\alpha \times \mathbb{R}$. Since the the transition functions

$$\begin{aligned} &\widetilde{\Phi}_{\alpha\mapsto\beta}\colon U_\alpha \cap U_\beta \to \mathbb{R}^*, \\ &(U_\alpha \cap U_\beta) \times \mathbb{R} \ni (x,v) \to (x, \widetilde{\Phi}_{\alpha\mapsto\beta}(x)v) \in (U_\alpha \cap U_\beta) \times \mathbb{R} \end{aligned}$$

preserve the standard metric on the trivial bundle $(U_\alpha \cap U_\beta) \times \mathbb{R}$, we must have $\widetilde{\Phi}_{\alpha\mapsto\beta} \equiv \pm 1$. □

**Exercise 16.4.** As before, fix a metric $g$ on $E$ and wtart with an arbitrary collection of local trivializations $\Phi_\alpha\colon E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^k$ over an open cover $\{U_\alpha\}$ of $M$. It is a result in linear algebra [**HJ13**, Chapter 7] that every semi-positive matrix admits a unique semi-positive square root. A continuous or smooth family of semi-positive matrices also admits a unique semi-continuous or smooth family of positive square roots. Let $\mathcal{G}_\alpha$ denote the semi-positive matrix-valued function of $g$ with respect to $\Phi_\alpha$ and

$$U_\alpha \times \mathbb{R}^k \longrightarrow U_\alpha \times \mathbb{R}^k, \qquad (x,v) \longrightarrow (x, \Theta_\alpha(x)v)$$

denote the bundle isomorphism corresponding to $\Theta_\alpha = \sqrt{\mathcal{G}_\alpha}$. Multiplying $\Phi_\alpha$ with $\Theta_\alpha$ defines a new collection of trivializations

$$\widetilde{\Phi}_\alpha = \Theta_\alpha \cdot \Phi_\alpha$$

that identifies $g|_{U_\alpha}$ with the standard metric on $U_\alpha \times \mathbb{R}^k$. Since the the transition functions

$$\widetilde{\Phi}_{\alpha\mapsto\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(n,\mathbb{R})$$

preserve the standard metric on the trivial bundle $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$, we must have $\widetilde{\Phi}_{\alpha\mapsto\beta} \in \mathrm{O}(k)$. Further, if the vector bundle is orientable, we can start from a collection of local trivializations compatible with the orientation and the modification above preserves this property. Therefore, $\widetilde{\Phi}_{\alpha\mapsto\beta} \in \mathrm{O}(k)$ and $\det(\widetilde{\Phi}_{\alpha\mapsto\beta}) > 0$ which implies $\widetilde{\Phi}_{\alpha\mapsto\beta} \in \mathrm{SO}(k)$ . □

**Exercise 16.6.** As we mentioned above, a Riemannian metric $g$ on a vector bundle $E$ yields an isomorphism between $E$ and $E^*$ by mapping

$$v \in E_p \longmapsto g(v, \cdot) \in E_p^* = \mathrm{Hom}(E_p, \mathbb{R}).$$

This isomorphism yields a metric on $E^*$ such that if $e_1, \ldots, e_k$ is an orthonormal basis for $E_p$ then the dual basis $e_1^*, \ldots, e_k^*$ is an orthonormal basis for $E_p^*$. These two bases induce bases for any tensorial product of $E$ and $E^*$ and thus define metrics on them for which the induced basis is orthonormal. Changing $e_1, \ldots, e_k$ to another orthonormal basis corresponds to multiplication by some $B \in \mathrm{O}(k)$. The induced bases also change by an orthogonal matrix. So the definition above is well-defined. □

**Exercise 16.8.** As we mentioned above, every two orthonormal frames are related by multiplication by a matrix valued function $B$ that takes values in $\mathrm{O}(r)$. Suppose $s_1, \ldots, s_r$ and $s_1', \ldots, s_r'$ are two (local) orthonormal frames for $(E, g)$. Then,

$$s_1' \wedge \ldots \wedge s_r' = \det(B)\ s_1 \wedge \ldots \wedge s_r = \pm s_1 \wedge \ldots \wedge s_r.$$

Moreover, if $E$ is oriented and both are positive frames in the sense of Definition 15.16, then $B \in \mathrm{SO}(r)$. □

**Exercise 16.10.** The chart maps are

$$\varphi_\pm \colon U_\pm := S^2 \setminus \{p_\pm\} \to \mathbb{R}^2, \qquad (y_1, y_2) = \varphi_\pm(x_0, x_1, x_2) = \frac{1}{1 \mp x_0}(x_1, x_2).$$

with the inverse

$$\begin{aligned} &\varphi_\pm^{-1} \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \\ &(y_1, y_2) \longrightarrow (x_0, x_1, x_2) = \frac{1}{y_1^2 + y_2^2 + 1} \left( \pm(y_1^2 + y_2^2 - 1), 2y_1, 2y_2 \right). \end{aligned}$$

In order to find the matrices of the induced metric on $S^2$ in each chart, we need to find the vector fields $d\varphi_\pm^{-1}(\partial_{y_1})$ and $d\varphi_\pm^{-1}(\partial_{y_2})$ and calculate their inner products. We have

$$\begin{aligned} \xi_1 = d\varphi_\pm^{-1}(\partial_{y_1}) &= \frac{1}{(y_1^2 + y_2^2 + 1)^2} \left( \pm 4y_1, 2 + 2(y_2^2 - y_1^2), -4y_1 y_2 \right), \\ \xi_2 = d\varphi_\pm^{-1}(\partial_{y_2}) &= \frac{1}{(y_1^2 + y_2^2 + 1)^2} \left( \pm 4y_2, -4y_1 y_2, 2 + 2(y_1^2 - y_2^2) \right). \end{aligned}$$

We conclude that

$$\begin{aligned} g_{22} = g_{11} =& \xi_1 \cdot \xi_1 \\ =& \frac{1}{(y_1^2+y_2^2+1)^4}\left(16y_1^2+4+4(y_2^2-y_1^2)^2+8(y_2^2-y_1^2)+16y_1^2y_2^2\right) \\ =& \frac{4}{(y_1^2+y_2^2+1)^4}(1+y_2^2+y_1^2)^2 = \frac{4}{(y_1^2+y_2^2+1)^2}, \\ g_{12} = g_{21} =& \xi_1 \cdot \xi_2 \\ =& \frac{\left(16y_1y_2-8y_1y_2+8y_1y_2(y_1^2-y_2^2)-8y_1y_2+8y_1y_2(y_2^2-y_1^2)\right)}{(y_1^2+y_2^2+1)^4} = 0; \end{aligned}$$

i.e. th matrices of the induced metric on $S^2$ in each chart are a multiple of the standard metric

$$\left[g_{ij}(y)\right] = \frac{4}{(y_1^2+y_2^2+1)^2}\mathrm{I}_2.$$

□

**Exercise 16.11.** The semicircle of radius $r$ centered at the origin can be parametrized by angle:

$$\gamma(\theta) = r(\cos(\theta), \sin(\theta)) \quad 0 < \theta < \pi.$$

We have

$$\dot{\gamma} := \frac{d\gamma}{d\theta} = r(-\sin(\theta)\partial_x + \cos(\theta)\partial_y).$$

Therefore,

$$|\dot{\gamma}|^2 = \frac{r^2\sin(\theta)^2 + r^2\cos(\theta)^2}{r^2\sin(\theta)^2} = \frac{1}{\sin(\theta)^2}.$$

We conclude that

$$|\gamma| = \int_0^\pi \frac{1}{\sin(\theta)^2}d\theta = \infty.$$

□

**Exercise 16.16.** With $z = x + \mathrm{i}y$, we have

$$dz = dx + \mathrm{i}dy.$$

Therefore,

$$dz \otimes \overline{dz} = (dx + \mathrm{i}dy) \otimes (dx - \mathrm{i}dy) = dx \otimes dx + dy \otimes dy.$$

Since $\mathrm{Im}(z) = y$, we get Show that the real part of the Hermitian metric

$$\mathrm{Re}\left(\frac{dz \otimes \overline{dz}}{\mathrm{Im}(z)^2}\right) = \frac{dx \otimes dx + dy \otimes dy}{y^2}.$$

Recall that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

acts on $\mathcal{H}$ by

$$z \longrightarrow \varphi_A(z) = \frac{az+b}{cz+d}.$$

Therefore,

$$d\varphi_A(\partial_z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2}\partial_z = (cz+d)^{-2}\partial_z.$$

We get

$$\mathfrak{h}\big(d\varphi_A(\partial_z), d\varphi_A(\partial_z)\big) = \frac{|cz+d|^{-4}}{\mathrm{Im}(\varphi_A(z))^2}.$$

We also have

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2}\right)$$
$$= \mathrm{Im}\left(\frac{ac|z|^2 + bd + adz + bc\overline{z}}{|cz+d|^2}\right) = \frac{y}{|cz+d|^2}.$$

Therefore,

$$\mathfrak{h}|_{\varphi_A(z)}\big(d\varphi_A(\partial_z), d\varphi_A(\partial_z)\big) = \frac{1}{y^2} = \mathfrak{h}|_z(\partial_z, \partial_z).$$

We conclude that the metric $\mathfrak{h}$ is preserved under the action of elements of $\mathrm{SL}(2,\mathbb{R})$. $\square$

**Exercise 16.17.** Fix a Hermitian metric $\mathfrak{h}$ on $E$. Given a collection of local trivializations

$$\Phi_\alpha \colon E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C}^k,$$

over an open cover $\{U_\alpha\}$ of $M$, let $\Phi_{\alpha\mapsto\beta} \in \mathrm{GL}(k,\mathbb{C})$ denote the change-of-trivialization matrix-valued functions of $E$, and let $H_\alpha$ denote the matrix form of $\mathfrak{h}$ with respect to $\Phi_\alpha$. Then,

$$H_\alpha = \Phi^T_{\alpha\mapsto\beta}\, H_\beta\, \overline{\Phi}_{\alpha\mapsto\beta} \qquad \forall\ \alpha,\beta.$$

Conversely, a collection $\{H_\alpha\}$ of positive-definite Hermitian $k\times k$ matrix-valued functions on $\{U_\alpha\}$ satisfying the compatibility relation above defines a well-defined Hermitian metric $\mathfrak{h}$ on $E$.

It is a result in linear algebra [**You88**, Section 7.4] that every (continuous or smooth family of) semi-positive Hermitian matrix admits a unique (continuous or smooth family of) semi-positive Hermitian square root. Let

$$U_\alpha \times \mathbb{C}^k \longrightarrow U_\alpha \times \mathbb{C}^k, \qquad (x,v) \longrightarrow (x, \overline{\Theta}_\alpha(x)v)$$

denote the linear transformation corresponding to $\Theta_\alpha = \sqrt{H_\alpha}$. Multiplying $\Phi_\alpha$ with $\overline{\Theta}_\alpha$ defines a new collection of trivializations

$$\widetilde{\Phi}_\alpha = \overline{\Theta}_\alpha \cdot \Phi_\alpha$$

that identifies $\mathfrak{h}|_{U_\alpha}$ with the standard Hermitian metric on $U_\alpha \times \mathbb{C}^k$. Since the the transition functions

$$\widetilde{\Phi}_{\alpha \mapsto \beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C})$$

preserve the standard Hermitian metric on the trivial bundle $(U_\alpha \cap U_\beta) \times \mathbb{C}^k$, we must have $\widetilde{\Phi}_{\alpha \mapsto \beta} \in \mathrm{U}(k)$. $\square$

Chapter 17

# Differential forms

For every vector space $V$, recall that the $k$-th exterior product $\Lambda^k V$ is defined as the quotient of

$$V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$$

by the subspace generated by differences of the form

$$v_1 \otimes \cdots \otimes v_k - \varepsilon(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \qquad \forall\ \sigma \in S_k,$$

where $\varepsilon(\sigma) \in \{\pm 1\}$ denotes the sign of the permutation $\sigma$. For $k, k' \geq 0$, the natural product map

$$V^{\otimes k} \otimes V^{\otimes k'} \longrightarrow V^{\otimes (k+k')}$$

descends to a wedge-product map

$$\Lambda^k V \otimes \Lambda^{k'} V \longrightarrow \Lambda^{k+k'} V$$

that sends $(v_1 \wedge \cdots \wedge v_k) \otimes (v_{k+1} \wedge \cdots \wedge v_{k+k'})$ to

$$v_1 \wedge \cdots \wedge v_k \wedge v_{k+1} \wedge \cdots \wedge v_{k+k'},$$

and extends linearly to arbitrary linear combinations of such generators. Likewise, for any vector bundle $E \longrightarrow M$, there are surjective wedge-product bundle homomorphisms

$$\Lambda^k E \otimes \Lambda^{k'} E \longrightarrow \Lambda^{k+k'} E. \tag{17.1}$$

For instance, if rank $E = r$, $E$ is orientable, and $k + k' = r$, then $\Lambda^r E$ is isomorphic to the trivial bundle $\mathcal{O} = M \times \mathbb{R}$ or $M \times \mathbb{C}$. Any choice of isomorphism $\Lambda^r E \longrightarrow \mathcal{O}$ then induces an isomorphism

$$\Lambda^k E \otimes \Lambda^{r-k} E \longrightarrow \mathcal{O}$$

that identifies $\Lambda^k E$ with the dual of $\Lambda^{r-k} E$.

**Exercise 17.1.** Let $V$ be an $n$-dimensional vector space $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Show that every $\alpha \in \Lambda^{n-1}V$ is the wedge of $n-1$ vectors; i.e.

$$\alpha = v_1 \wedge \cdots \wedge v_{n-1}, \quad v_i \in V \ \forall\, i = 1, \ldots, n-1.$$

For $n = 4$, give an example of $\alpha \in \Lambda^2\mathbb{R}^4$ that can cannot be written as $v_1 \wedge v_2$.

**Exercise 17.2.** Let $\eta \in \Lambda^2 V$ where $V$ is some vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Show that there exists a basis $e_1, e_2, \ldots.$ for $V$ such that

$$\eta = e_1 \wedge e_2 + \cdots + e_{2r-1} \wedge e_{2r}$$

for some $r \geq 0$.

The exterior products of the cotangent bundle play a central role in differential geometry, both in the smooth and holomorphic settings. Given a smooth or holomorphic manifold $M$, sections of $\Lambda^k T^*M$ are called differential $k$-forms and are fundamental objects in modern geometry and mathematical physics: they provide a coordinate-free language for multivariable calculus, encode topological invariants through de Rham cohomology, and generalize holomorphic functions in the complex analytic context. In the holomorphic category, the *sheaves* of holomorphic differential forms carry rich algebraic and geometric information, playing a key role in Hodge theory and the study of complex and Kähler manifolds. In this section, we delve into these definitions. The wedge product defined above turns the space of differential forms into a non-commutative algebra.

**Definition 17.3.** Given a smooth manifold $M$, a **differential $k$-form** on $M$ is a section of $\Lambda^k T^*M$. The space of differential $k$-forms on $M$ will be denoted by $\Omega^k(M)$.

If $\varphi \colon U \to V \subset \mathbb{R}^m$ is a chart with coordinates $(x_1, \ldots, x_m)$ on $\mathbb{R}^m$, then $dx_1, \ldots, dx_m$ form the natural frame for $T^*M|_U = T^*V$, and every $k$-form $\eta$ on $V$ can be written as

$$\eta = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where the coefficients $a_{i_1 \cdots i_k}$ are smooth functions on $V$.

Globally, given an atlas

$$\mathcal{A} = \{\varphi_\alpha \colon U_\alpha \to V_\alpha\}$$

on $M$, a differential $k$-form $\eta$ on $M$ corresponds, by (12.1), to a collection of local $k$-forms $\eta_\alpha$ on $V_\alpha$ satisfying the compatibility condition

$$\eta_\alpha|_{V_{\beta,\alpha}} = \varphi^*_{\alpha\mapsto\beta}\left(\eta_\beta|_{V_{\alpha,\beta}}\right), \tag{17.2}$$

on the overlap $V_{\alpha,\beta} = \varphi_\alpha(U_\alpha \cap U_\beta) = V_\alpha \cap \varphi^{-1}_{\alpha\mapsto\beta}(V_\beta)$.

Here, the **pullback map**

$$\varphi^*_{\alpha\mapsto\beta}\colon \Omega^k(V_\beta) \longrightarrow \Omega^k(V_\alpha)$$

is induced by the pullback of 1-forms. If $x = (x_1, \ldots, x_m)$ are coordinates on $V_\alpha$ and $y = (y_1, \ldots, y_m)$ are coordinates on $V_\beta$, with $y = y(x) = \varphi_{\alpha\mapsto\beta}(x)$ on the overlap, then $\eta_\beta$ has the form

$$\eta_\beta = \sum_{i_1<\cdots<i_k} b_{i_1\cdots i_k}(y)\, dy_{i_1} \wedge \cdots \wedge dy_{i_k},$$

and the pullback is given by

$$\varphi^*_{\alpha\mapsto\beta}(\eta_\beta) = \sum_{i_1<\cdots<i_k} b_{i_1\cdots i_k}(y(x))\, dy_{i_1}(x) \wedge \cdots \wedge dy_{i_k}(x).$$

To express the right-hand side in terms of the basis $dx_j$, one applies the chain rule to each 1-form $dy_j$, expanding it as

$$dy_j(x) = \sum_{i=1}^m \frac{\partial y_j}{\partial x_i}\, dx_i.$$

**Example 17.4.** For $k = m = \dim M$, $\eta_\beta = b(y)dy_1 \wedge \cdots \wedge dy_m$, $\eta_\alpha = a(x)dx_1 \wedge \cdots \wedge dx_m$ and the compatibility relation reads

$$\begin{aligned} &\varphi^*_{\alpha\mapsto\beta}(\eta_\beta) = b(y(x))\, dy_1(x) \wedge \cdots \wedge dy_m(x) \\ &\ = b(y(x)) \det(d\varphi_{\alpha\mapsto\beta}) dx_1 \wedge \cdots \wedge dx_m = a(x)\, dx_1 \wedge \cdots \wedge dx_m. \end{aligned}$$

Therefore,

$$a(x) = b(y(x)) \det(d\varphi_{\alpha\mapsto\beta}).$$

For $k = 0$, $\Lambda^0 T^*M = M \times \mathbb{R}$; therefore, a differential 0-form is nothing but a smooth function on $M$. For $k = 1$, as we briefly studied earlier, a differential 1-form is a section of cotangent bundle $T^*M$. The duality pairing

$$\Gamma(M, TM) \otimes \Omega^1(M) \longrightarrow C^\infty(M, \mathbb{R})$$

between smooth vector fields and differential 1-forms takes as input a pair of a vector field $\xi$ and a differential form $\eta$ and returns a function $\eta(\xi)$ obtained by point-wise action of $\eta$ on $\xi$. More generally, we have the following.

For every $k \geq 1$, there is a natural degree-decreasing pairing

$$\Gamma(M, TM) \otimes \Omega^k(M) \longrightarrow C^{k-1}(M, \mathbb{R})$$

called **contraction by a vector field**, denoted by

$$\xi \otimes \eta \longmapsto \iota_\xi \eta$$

between every vector field $\xi \in \Gamma(M, TM)$ and $k$-form $\eta \in \Omega^k(M)$.

More precisely, every $\eta \in \Omega^k(M)$ is a pointwise skew-symmetric $k$-linear map on the tangent bundle, and $\iota_\xi \eta$ is defined by inserting $\xi$ as the first input. In other words, for every $\zeta_1, \dots, \zeta_{k-1} \in \Gamma(M, TM)$, the $(k-1)$-form $\iota_\xi \eta$ acts as

$$(\iota_\xi \eta)(\zeta_1, \dots, \zeta_{k-1}) = \eta(\xi, \zeta_1, \dots, \zeta_{k-1}).$$

By (17.1), there is a wedge product map

$$\Omega^k(M) \otimes \Omega^{k'}(M) \longrightarrow \Omega^{k+k'}(M)$$

that is locally given by

$$\left(\sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \otimes \left(\sum_{j_1<\cdots<j_{k'}} b_{j_1\cdots j_{k'}}\, dx_{j_1} \wedge \cdots \wedge dx_{j_{k'}}\right) \mapsto$$
$$\sum_{i_1<\cdots<i_k} \sum_{j_1<\cdots<j_{k'}} a_{i_1\cdots i_k} b_{j_1\cdots j_{k'}}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k'}}.$$

Of course, if the index sets $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_{k'}\}$ have an index in common, then

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{k'}} = 0.$$

Otherwise, one can reorder the union into increasing order at the cost of possibly introducing a sign.

Also, with the same notation, the contraction between

$$\xi = \sum_{j=1}^{m} b_j\, \partial_{x_j}$$

and

$$\eta = \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

returns

$$\iota_\xi \eta = \sum_{i_1<\cdots<i_k} \sum_{c=1}^{k} (-1)^{c-1} b_{i_c} a_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_{c-1}} \wedge dx_{i_{c+1}} \wedge \cdots \wedge dx_{i_k}.$$

The reason for the sign $(-1)^{c-1}$ is that we first move $dx_{i_c}$ to the first position before evaluating it on $b_{i_c}\, \partial_{x_{i_c}}$. This requires commuting past $c-1$ differentials, introducing a sign of $(-1)^{c-1}$.

Note that, since every differential $k$-form is a skew-symmetric $k$-linear map on sections of the tangent bundle, we have

$$\iota_\xi \circ \iota_\xi = 0.$$

Among all differential forms on a smooth manifold, those of top degree – i.e., forms of degree equal to the dimension of the manifold – play a distinguished role in extending the notions of classical calculus to the setting of manifolds. These forms are precisely the ones that can be integrated over oriented manifolds and thus serve as the foundation for a coordinate-free formulation of integration theory, culminating in powerful generalizations such as Stokes' theorem.

**Definition 17.5.** A **volume form** on a smooth $m$-manifold $M$ is a differential form $\omega \in \Omega^{\text{top}}(M) := \Omega^m(M)$ that is nowhere vanishing. In other words, a volume form is a nowhere zero section of $\Lambda^{\text{top}}(T^*M)$.

Since there is a one-to-one correspondence between nowhere vanishing sections of a line bundle and its trivializations, a smooth manifold $M$ admits a volume form if and only if $\Lambda^{\text{top}}(T^*M) \cong M \times \mathbb{R}$. As $\Lambda^{\text{top}}(T^*M)$ is the dual of $\Lambda^{\text{top}}(TM)$, the triviality of one implies the triviality of the other. Hence, $M$ admits a volume form if and only if it is orientable.

Furthermore, there is a one-to-one correspondence between volume forms and isomorphisms $\det(T^*M) = \Lambda^{\text{top}}(T^*M) \cong M \times \mathbb{R}$, and between orientations on $M$ and volume forms up to multiplication by a positive function.

If $\varphi\colon U \to V \subset \mathbb{R}^m$ is a chart with coordinates $(x_1, \ldots, x_m)$ on $\mathbb{R}^m$, then every $m$-form $\omega$ admits a local expression

$$\omega|_U = f(x)\, dx_1 \wedge \cdots \wedge dx_m$$

for some smooth function $f(x)$. If $\omega$ is a volume form, then $f(x)$ is nowhere zero. If $M$ is oriented and $\varphi$ belongs to the oriented atlas of $M$, then a volume form is compatible with the orientation if and only if $f(x) > 0$.

Once again, an orientation corresponds to a choice of trivialization

$$\Lambda^{\text{top}}(T^*M) \cong M \times \mathbb{R}$$

up to rescaling by a positive function, and a volume form compatible with the orientation corresponds to a positive multiple of the constant section 1 of $M \times \mathbb{R}$.

**Proposition 17.6.** *Suppose $M$ is an oriented smooth manifold and $\mathrm{g}$ is a Riemannian metric on $M$. If $\varphi\colon U \to V \subset \mathbb{R}^m$ is a chart with coordinates $(x_1, \ldots, x_m)$ on $\mathbb{R}^m$, let $\big[\mathrm{g}_{ij}(x)\big]$ denote the positive-definite matrix representing the metric $\mathrm{g}$ in these coordinates. Define the local $m$-form*

$$\omega_{\mathrm{g},\varphi} = \sqrt{\det[\mathrm{g}_{ij}(x)]}\, dx_1 \wedge \cdots \wedge dx_m.$$

*As $\varphi$ varies over the positively oriented charts of $M$, the locally defined forms $\omega_{\mathrm{g},\varphi}$ agree on chart overlaps and therefore assemble into a global volume form* *on $M$, denoted by $\omega_g$, which is canonically associated to the metric $g$ and the chosen orientation on $M$.*

*In other words, every Riemannian metric on an oriented manifold determines a canonical volume form.*

**Proof.** Suppose $\varphi_1 \colon U_1 \to V_1 \subset \mathbb{R}^m$ and $\varphi_2 \colon U_2 \to V_2 \subset \mathbb{R}^m$ are two overlapping charts with coordinates $(x_1, \ldots, x_m)$ on $V_1$ and $(y_1, \ldots, y_m)$ on $V_2$, respectively.

By (17.2), the local volume forms $\omega_{g,\varphi_1}$ and $\omega_{g,\varphi_2}$ are compatible on the overlap if and only if

$$\omega_{g,\varphi_1} = \varphi_{1\mapsto 2}^*(\omega_{g,\varphi_2}).$$

By (16.1), the transformation rule for the metric tensor gives:

$$\det[g_{ij}(x)] = \det\left(d\varphi_{1\mapsto 2}^T\,[g_{ij}(y)]\,d\varphi_{1\mapsto 2}\right) = \det[g_{ij}(y)] \cdot \det(d\varphi_{1\mapsto 2})^2.$$

Also, by (14.3), the pullback of the standard volume form transforms as:

$$\varphi_{1\mapsto 2}^*(dy_1 \wedge \cdots \wedge dy_m) = \det(d\varphi_{1\mapsto 2})\, dx_1 \wedge \cdots \wedge dx_m. \tag{17.3}$$

Since both charts belong to the positively oriented atlas of $M$, we have $\det(d\varphi_{1\mapsto 2}) > 0$. Therefore, the pullback of $\omega_{g,\varphi_2}$ under $\varphi_{1\mapsto 2}$ is computed as:

$$\begin{aligned}
\varphi_{1\mapsto 2}^*(\omega_{g,\varphi_2}) &= \varphi_{1\mapsto 2}^*\left(\sqrt{\det[g_{ij}(y)]}\, dy_1 \wedge \cdots \wedge dy_m\right) \\
&= \sqrt{\det[g_{ij}(y(x))]} \cdot \varphi_{1\mapsto 2}^*(dy_1 \wedge \cdots \wedge dy_m) \\
&= \sqrt{\frac{\det[g_{ij}(x)]}{\det(d\varphi_{1\mapsto 2})^2}} \cdot \det(d\varphi_{1\mapsto 2})\, dx_1 \wedge \cdots \wedge dx_m \\
&= \sqrt{\det[g_{ij}(x)]}\, dx_1 \wedge \cdots \wedge dx_m \\
&= \omega_{g,\varphi_1}.
\end{aligned}$$

Thus, the local expressions $\omega_{g,\varphi}$ agree on overlaps, and define a global smooth volume form on $M$. $\square$

The proposition above provides a method for finding volume forms on any oriented manifold. The following result serves the same purpose for manifolds realized as level sets of smooth functions in ambient manifolds equipped with a natural volume form, such as $\mathbb{R}^m$ with the standard volume form $\omega_{\text{std}} = dx_1 \wedge \cdots \wedge dx_m$.

**Lemma 17.7.** *Suppose $f \colon M \to \mathbb{R}$ is a smooth function on a manifold equipped with a volume form $\omega_M$, $q \in \mathbb{R}$ is a regular value, and $Y = f^{-1}(q) \subset M$ is the corresponding level set. Also, suppose $\xi$ is a vector field defined on a neighborhood of $Y$ (or simply a section of $TM|_Y$) such that $\xi$ is not* *tangent to $Y$ along $Y$. Then the restriction $\omega_Y$ of $\iota_\xi \omega_M$ to $Y$ is a volume form on $Y$.*

**Proof.** Since $\xi$ is not tangent to $Y$ along $Y$, we have

$$TM|_Y = TY \oplus \mathbb{R} \cdot \xi.$$

Therefore, if $\dim M = m$, then for every $p \in Y$ and every frame $v_1, \ldots, v_{m-1}$ for $T_pY$, the tuple

$$(\xi(p), v_1, \ldots, v_{m-1})$$

is a frame for $T_pM$. By the definition of a volume form, we have

$$\omega_M|_p(\xi(p), v_1, \ldots, v_{m-1}) \neq 0.$$

It follows that

$$\omega_Y|_p(v_1, \ldots, v_{m-1}) = (\iota_\xi \omega_M)|_p(v_1, \ldots, v_{m-1}) = \omega_M|_p(\xi(p), v_1, \ldots, v_{m-1}) \neq 0.$$

Therefore, $\omega_Y$ is a volume form on $Y$. □

The construction above requires a vector field that is not tangent to $Y$ along $Y$. A natural way to obtain such a vector field is by considering the gradient vector field of $f$ with respect to some Riemannian metric on $M$; see (16.2). This is particularly straightforward when $M = \mathbb{R}^m$ with the standard metric. We can further normalize $\nabla f$ to

$$\vec{n} = \frac{\nabla f}{|\nabla f|}$$

to obtain a unit-length vector field that is orthogonal to $TY$. For this orthonormal vector field, we have the following.

**Exercise 17.8.** Suppose $M$ is an oriented manifold equipped with a Riemannian metric $g$. Let $\omega$ denote the volume form of $g$ in the sense of Proposition 17.6. Suppose $f\colon M \to \mathbb{R}$ is a smooth function, $q \in \mathbb{R}$ is a regular value, and $Y = f^{-1}(q) \subset M$ is the corresponding level set. Let $g_Y$ denote the induced metric on $Y$, and let $\omega_Y$ denote the volume form of $g_Y$. Also, let

$$\vec{n} = \frac{\nabla f}{|\nabla f|}, \quad \text{and} \quad \omega'_Y = \iota_{\vec{n}} \omega$$

denote the volume form on $Y$ induced via Lemma 17.7. Show that

$$\omega_Y = \pm \omega'_Y,$$

where the sign depends on the convention for the induced orientation on $Y$ and the direction of $\vec{n}$.

**Example 17.9.** Consider the unit sphere $S^2 = f^{-1}(1) \subset \mathbb{R}^3$, where $f(x, y, z) = x^2 + y^2 + z^2$. The gradient vector field of $f$ with respect to the standard metric is

$$\nabla f = 2(x\partial_x + y\partial_y + z\partial_z),$$

which gives the orthonormal vector field

$$x\partial_x + y\partial_y + z\partial_z$$

along $S^2$. The volume form on $\mathbb{R}^3$ with respect to the standard metric is simply

$$dx \wedge dy \wedge dz.$$

Therefore, the induced volume form (area form) on $S^2$ is the restriction of the 2-form

$$\iota_{x\partial_x+y\partial_y+z\partial_z}(dx \wedge dy \wedge dz) = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

to $S^2$.

**Exercise 17.10.** Let $\omega_m$ denote the volume form of the induced metric on $S^m$ defined by embedding $S^m$ as the unit sphere in $\mathbb{R}^{m+1}$. Show that on each open hemisphere $x_0 \neq 0$, this volume form coincides with the restriction of

$$\frac{1}{x_0} dx_1 \wedge \ldots \wedge dx_m$$

to $S^m$.

**Exercise 17.11.** Let $M$ be a smooth orientable $m$-manifold, and suppose that $\omega$ is a volume-form. Show that every point of $M$ is included in a chart with coordinates $(x_1, x_2, \ldots, x_m)$ such that $\omega = dx_1 \wedge \ldots \wedge dx_m$. Use this to prove that a smooth manifold is orientable iff it admits a smooth atlas whose coordinate transition functions $\varphi\colon (x_1, x_2, \ldots, x_m) \longrightarrow (y_1, y_2, \ldots, y_m)$ all satisfy $\det d\varphi \equiv 1$.

## Solutions to exercises

**Exercise 17.1.** Wedging with $\alpha$ defines a non-trivial linear map

$$V \longrightarrow \mathbb{F}, \quad v \longmapsto \alpha \wedge v \in \Lambda^n V \cong \mathbb{F}.$$

The kernel of this map is an $(n-1)$-dimensional subspace of $V$ generated by a set of $(n-1)$ vectors $v_1, \ldots, v_{n-1}$. Choose $v_n \in V$ such that $\alpha \wedge v_n \neq 0$. Clearly, $v_1, \ldots, v_n$ is a basis for $V$. Thus, $v_1 \wedge \cdots \wedge v_n$ generates $\Lambda^n V$.

Suppose

$$\alpha \wedge v_n = \lambda\, v_1 \wedge \cdots \wedge v_n$$

for some $\lambda \neq 0$. Then, both $\alpha$ and $\lambda\, v_1 \wedge \cdots \wedge v_{n-1}$ define the same linear maps on $V$, and therefore they must be equal. We conclude that

$$\alpha = (\lambda v_1) \wedge v_2 \wedge \cdots \wedge v_{n-1}$$

is a wedge of $(n-1)$ vectors.

For $V = \mathbb{R}^4$, let

$$\alpha = e_1 \wedge e_2 + e_3 \wedge e_4 \in \Lambda^2 \mathbb{R}^4.$$

Wedging with $\alpha$ defines a linear map

$$\mathbb{R}^4 \longrightarrow \Lambda^3 \mathbb{R}^4 \cong \mathbb{R}^4$$

given by

$$\begin{aligned}
e_1 &\longmapsto e_1 \wedge \alpha = u_2 := e_1 \wedge e_3 \wedge e_4, \\
e_2 &\longmapsto e_2 \wedge \alpha = u_1 := e_2 \wedge e_3 \wedge e_4, \\
e_3 &\longmapsto e_3 \wedge \alpha = u_4 := e_1 \wedge e_2 \wedge e_3, \\
e_4 &\longmapsto e_4 \wedge \alpha = u_3 := e_1 \wedge e_2 \wedge e_4.
\end{aligned}$$

The matrix of this linear map with respect to the basis $(e_1, e_2, e_3, e_4)$ on the domain and $(u_1, u_2, u_3, u_4)$ on the target is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which has rank 4.

On the other hand, any decomposable element $v_1 \wedge v_2 \in \Lambda^2 \mathbb{R}^4$ defines a linear map

$$\mathbb{R}^4 \longrightarrow \Lambda^3 \mathbb{R}^4 \cong \mathbb{R}^4$$

of rank 2. Therefore, $\alpha$ cannot be written as $v_1 \wedge v_2$. $\square$

**Exercise 17.2.** Suppose $e_1, e_2, \ldots$ is an arbitrary basis for $V$. Then, every element $\eta \in \Lambda^2 V$ is a linear combination

$$\eta = \sum_{i<j} a_{ij}\, e_i \wedge e_j.$$

To write this symmetrically – and since $u \wedge w = -w \wedge u$ for all $u, w \in V$– we can express $\eta$ in the form

$$\eta = \sum_{i,j} b_{ij}\, e_i \wedge e_j,$$

where $b_{ii} = 0$ and $b_{ij} = -b_{ji} = a_{ij}/2$ for all $i < j$. In other words, once a basis is fixed, there is a one-to-one correspondence between elements $\eta \in \Lambda^2 V$ and skew-symmetric matrices $B = [b_{ij}]$.

If $e_1', e_2', \ldots$ is another basis, with change of basis matrix $\Theta = [\theta_{ij}]$ such that

$$e_j' = \sum_i \theta_{ji}\, e_i,$$

then the skew-symmetric matrices $B$ and $B'$ corresponding to the bases $\{e_i\}$ and $\{e_i'\}$ are related by

$$B' = \Theta^T B \Theta.$$

Therefore, Exercise 17.2 is equivalent to showing that for every skew-symmetric matrix $B$, there exists an invertible matrix $\Theta$ such that $\Theta^T B \Theta$ is of the form

$$\begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & 0 & \cdots & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The latter is a classical result in linear algebra; see [**Lan89**, Theorem 4.4]. □

**Exercise 17.8.** Suppose $M$ is an oriented manifold equipped with a Riemannian metric $g$. Let $\omega$ denote the volume form of $g$ in the sense of Proposition 17.6. Suppose $f\colon M \to \mathbb{R}$ is a smooth function, $q \in \mathbb{R}$ is a regular value, and $Y = f^{-1}(q) \subset M$ is the corresponding level set. Let $g_Y$ denote the induced metric on $Y$, and let $\omega_Y$ denote the volume form of $g_Y$. Also, let

$$\vec{n} = \frac{\nabla f}{|\nabla f|}, \quad \text{and} \quad \omega_Y' = \iota_{\vec{n}}\omega$$

denote the volume form on $Y$ induced via Lemma 17.7. Show that

$$\omega_Y = \pm \omega_Y',$$

where the sign depends on the convention for the induced orientation on $Y$ and the direction of $\vec{n}$.

**Exercise 17.10.** The hemisphere $S^m \cap (x_0 > 0)$ is the graph of

$$x_0 = \varphi(x_1, \ldots, x_m) = \sqrt{1 - \sum_{i=1}^{m} x_i^2}$$

over the unit ball $B_1(0) \subset \mathbb{R}^m$.

Similarly to Example 17.9, the volume form $\omega_m$ is obtained by contracting the standard volume form $dx_0 \wedge \cdots \wedge dx_m$ on $\mathbb{R}^{m+1}$ with the vector field

$$x_0 \partial_{x_0} + \cdots + x_m \partial_{x_m},$$

and restricting the resulting $m$-form to $S^m$; that is,

$$\omega_m = \left( \sum_{i=0}^{m} (-1)^i x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m \right) \Bigg|_{S^m}.$$

The question asks us to show that this coincides with

$$\left( \frac{1}{x_0} \, dx_1 \wedge \cdots \wedge dx_m \right) \Bigg|_{S^m}.$$

To show this, we compute the pullbacks of both forms by $\varphi$ and verify they agree on the domain $B_1(0)$ of the chart.

We have

$$\varphi^* dx_0 = d\varphi = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i} dx_i = -\sum_{i=1}^{m} \frac{x_i}{\varphi} dx_i = -\sum_{i=1}^{m} \frac{x_i}{x_0} dx_i.$$

Therefore,

$$\begin{aligned} &\varphi^* \left( \sum_{i=0}^{m} (-1)^i x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m \right) \\ &= \frac{1}{x_0} \left( x_0^2 \, dx_1 \wedge \cdots \wedge dx_m + \sum_{i=1}^{m} x_i^2 \, dx_1 \wedge \cdots \wedge dx_m \right) \\ &= \frac{1}{x_0} \, dx_1 \wedge \cdots \wedge dx_m. \end{aligned}$$

The computation on the other half is similar. □

**Exercise 17.11.** Suppose $\varphi \colon U \longrightarrow V \subset \mathbb{R}^m$ is a chart compatible with the orientation. With respect to the local coordinates $(x_1, \ldots, x_m)$ on $V$ we

have
$$\omega|_U = f(x)dx_1 \wedge \cdots \wedge dx_m$$
for some positive function $f$. At the cost of shrinking $U$ and $V$ we may assume $V = (-\epsilon, \epsilon) \times V'$ for some open subset $V' \subset \mathbb{R}^{m-1}$. For every $x' = (x_2, \ldots, x_m) \in V'$ let
$$y_1(x_1, x') = \int_0^{x_1} f(t, x')dt.$$
Since $f > 0$, $y_1$ is an increasing function of $x_1$. Therefore, the composition
$$U \xrightarrow{\varphi} (-\epsilon, \epsilon) \times V' \xrightarrow{(x_1,x')\to(y_1,x')} \mathbb{R}^m$$
define a new chart map $\widetilde{\varphi}\colon U \longrightarrow \mathbb{R}^m$ with respect to which
$$\omega|_U = dy_1 \wedge dx_2 \wedge \cdots \wedge dx_m.$$
Let $M$ be a smooth orientable $m$-manifold, and suppose that $\omega$ is a volume-form. Show that every point of $M$ is included in a chart with coordinates $(x_1, x_2, \ldots, x_m)$ such that $\omega = dx_1 \wedge \ldots \wedge dx_m$.

Covering $M$ with a collection of such charts $\{\varphi_\alpha\colon U_\alpha \longrightarrow V_\alpha\}$, it is clear from Example 17.4 that the transition functions $\varphi_{\alpha\mapsto\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfy $\det d\varphi_{\alpha\mapsto\beta} \equiv 1$. By Proposition 15.8, the converse holds as well. $\square$

Chapter 18

# Exterior derivative and cohomology

The **exterior derivative** is a fundamental operator in differential geometry that extends the concept of differentiation to differential forms on smooth manifolds. It takes a $k$-form to a $(k+1)$-form in a way that generalizes classical notions such as the gradient, curl, and divergence from vector calculus. Defined intrinsically and without reliance on coordinates, the exterior derivative $d$ is linear, satisfies the graded Leibniz rule with respect to the wedge product, and is nilpotent: $d^2 = 0$. Notably, such a canonical differential operator does not exist on the exterior powers of the tangent bundle or on arbitrary tensor fields; defining a derivation in those contexts typically requires additional geometric structure (such as a connection). In contrast, the cotangent bundle and its exterior algebra – i.e., the differential forms – admit a natural and elegant differential calculus, making them more flexible and powerful tools for encoding geometry and topology.

Recall that given a smooth function $f\colon M \longrightarrow \mathbb{R}$, the derivative of $f$ can be seen as a differential 1-form that in local coordinates takes the form

$$df = \sum \frac{\partial f}{\partial x_i} dx_i. \tag{18.1}$$

The operator $d\colon \Omega^0(M) \longrightarrow \Omega^1(M)$ locally defined as above is globally well-defined because of the chain rule and how a collection of local differential forms define a global form in (17.2). The same reasoning extends to all differential forms and yields **exterior differentiation** maps

$$d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

for all $k \geq 0$. More precisely, first, we prove the following lemma.

**Definition 18.1.** Given an open subset $V \subset \mathbb{R}^m$ and any $k$-form

$$\eta = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

on $V$, the exterior derivative of $\eta$ is the $(k+1)$-form

$$d\eta := \sum_{i_1 < \cdots < i_k} da_{i_1 \cdots i_k}(x) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $da_{i_1 \cdots i_k}(x)$ should be expanded as in (18.1). In particular, $d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = 0$.

**Lemma 18.2.** *Pullback of differential forms by any smooth map*

$$f \colon V \subset \mathbb{R}^m \longrightarrow V' \subset \mathbb{R}^n$$

*commutes with $d$; i.e. $f^* \circ d = d \circ f^*$.*

**Proof.** Let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ denote the coordinates on the domain and target, respectively, with $y = y(x) = f(x)$. Since both $f^*$ and $d$ are $\mathbb{R}$-linear, it is enough to confirm the claim on a single term

$$\eta = b(y)\, dy_{i_1} \wedge \cdots \wedge dy_{i_k}.$$

We have

$$f^*\eta = b(y(x))\, dy_{i_1}(x) \wedge \cdots \wedge dy_{i_k}(x) = b(y(x)) \sum_{j_1, \ldots, j_k} \frac{\partial y_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial y_{i_k}}{\partial x_{j_k}}\, dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

where the sum runs over all tuples $(j_1, \ldots, j_k)$ with distinct indices $j_a \neq j_b$ for $a \neq b$.

Therefore, using the product rule,

$$\begin{aligned}
df^*\eta &= \sum_{j_1, \ldots, j_k} d\left( b(y(x)) \frac{\partial y_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial y_{i_k}}{\partial x_{j_k}} \right) dx_{j_1} \wedge \cdots \wedge dx_{j_k} \\
&= d\, b(y(x)) \wedge \sum_{j_1, \ldots, j_k} \frac{\partial y_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial y_{i_k}}{\partial x_{j_k}}\, dx_{j_1} \wedge \cdots \wedge dx_{j_k} \\
&\quad + b(y(x)) \sum_{j_1, \ldots, j_k} d\left( \frac{\partial y_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial y_{i_k}}{\partial x_{j_k}} \right) dx_{j_1} \wedge \cdots \wedge dx_{j_k} \\
&= d\, b(y(x)) \wedge dy_{i_1}(x) \wedge \cdots \wedge dy_{i_k}(x) \\
&\quad + b(y(x)) \sum_{j_0, j_1, \ldots, j_k} \frac{\partial \left( \frac{\partial y_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial y_{i_k}}{\partial x_{j_k}} \right)}{\partial x_{j_0}} dx_{j_0} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}.
\end{aligned}$$

On the other hand,

$$d\eta = d\, b(y) \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_k}$$

and

$$f^* d\eta = d\, b(y(x)) \wedge dy_{i_1}(x) \wedge \cdots \wedge dy_{i_k}(x).$$

Therefore, to prove the lemma, it suffices to show that

$$\sum_{j_0, j_1, \ldots, j_k} \frac{\partial \left( \frac{\partial y_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial y_{i_k}}{\partial x_{j_k}} \right)}{\partial x_{j_0}} dx_{j_0} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k} = 0.$$

By the product rule, we can expand the expression as

$$\sum_{j_0, j_1, \ldots, j_k} \sum_{a=1}^{k} \frac{\partial^2 y_{i_a}}{\partial x_{j_0} \partial x_{j_a}} \left( \prod_{c \neq a} \frac{\partial y_{i_c}}{\partial x_{j_c}} \right) dx_{j_0} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

Now observe that switching $j_0$ and $j_a$ keeps $\frac{\partial^2 y_{i_a}}{\partial x_{j_0} \partial x_{j_a}}$ unchanged but flips the sign of the wedge product. Therefore, each term in the sum appears with equal magnitude and opposite sign, so the total sum vanishes. □

**Lemma 18.3.** *The operator $d$ in Definition 18.1 satisfies $d \circ d = 0$ and*

$$d(\eta_1 \wedge \eta_2) = d\eta_1 \wedge \eta_2 + (-1)^{\deg(\eta_1)} \eta_1 \wedge d\eta_2.$$

**Proof.** The proof of the first statement is similar to the vanishing argument above. We have

$$\begin{aligned} d \circ d\,(a(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) &= d \sum_{i_0} \frac{\partial a(x)}{\partial x_{i_0}} dx_{i_0} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{i_0, i_0'} \frac{\partial^2 a(x)}{\partial x_{i_0'} \partial x_{i_0}} dx_{i_0'} \wedge dx_{i_0} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned}$$

Switching $i_0$ and $i_0'$ keeps the second derivative term unchanged but negates the wedge product. Hence the terms in the double sum cancel in pairs.

For the second statement, apply the product rule. To apply $d$ to $\eta_2$, we must move it before $\eta_1$ and then return $d\eta_2$ to the correct position. The first step introduces $\deg(\eta_1)\deg(\eta_2)$ transpositions, and the second introduces $\deg(\eta_1)(\deg(\eta_2) + 1)$. Thus, the total sign is

$$(-1)^{2\deg(\eta_1)\deg(\eta_2) + \deg(\eta_1} = (-1)^{\deg(\eta_1)}.$$

□

**Corollary 18.4.** *Given a smooth manifold $M$, for every $k \geq 0$, there is a global exterior differentiation map*

$$d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M) \tag{18.2}$$

*such that:*

(1) *locally in every chart it is given by Definition 18.1;*

(2) $d \circ d = 0$;

(3) $d(\eta_1 \wedge \eta_2) = d\eta_1 \wedge \eta_2 + (-1)^k \eta_1 \wedge d\eta_2$ *for all* $\eta_1 \in \Omega^k(M)$ *and* $\eta_2 \in \Omega^\ell(M)$;

(4) *if* $f\colon M \to N$ *is a smooth map between manifolds, then* $f^* \circ d = d \circ f^*$.

**Proof.** Recall from (17.2) that given an atlas

$$\mathcal{A} = \{\varphi_\alpha\colon U_\alpha \to V_\alpha\}$$

on $M$, a differential $k$-form $\eta$ on $M$ corresponds, by (12.1), to a collection of local $k$-forms $\eta_\alpha$ on $V_\alpha$ satisfying the compatibility condition

$$\eta_\alpha|_{V_{\beta,\alpha}} = \varphi^*_{\alpha\mapsto\beta}\left(\eta_\beta|_{V_{\alpha,\beta}}\right), \tag{18.3}$$

on the overlaps $V_{\alpha,\beta}$. By Lemma 18.2, for every $\alpha$ and $\beta$ we have

$$d\eta_\alpha = d\left(\varphi^*_{\alpha\mapsto\beta}\eta_\beta\right) = \varphi^*_{\alpha\mapsto\beta}\left(d\eta_\beta\right)$$

on the overlaps. Therefore, the locally defined exterior derivatives $d$ are compatible and define a global exterior differentiation map

$$d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M).$$

Item 1 holds by construction. Items 2–4 are local properties and thus follow from the two lemmas above. □

Considering the operators $d$ in (18.2) for all $k \geq 0$ results in a sequence

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \longrightarrow \Omega^{\dim M}(M) \xrightarrow{d} 0 \tag{18.4}$$

that is an example of a cochain complex over the field of real numbers.

**Definition 18.5.** Suppose $\{A_k\}$ is a collection of abelian groups or vector spaces, and

$$\cdots \longrightarrow A_{k-1} \xrightarrow{d} A_k \xrightarrow{d} A_{k+1} \longrightarrow \cdots$$

is a sequence of additive (or linear) maps between them. We say this is a **cochain complex** if $d \circ d = 0$. The **cohomology groups** of a cochain complex $(A_\bullet, d)$ are the quotient abelian groups or vector spaces

$$H^k(A_\bullet, d) = \frac{\ker(d\colon A_k \to A_{k+1})}{\operatorname{Im}(d\colon A_{k-1} \to A_k)}.$$

An element in the kernel of $d\colon A_k \to A_{k+1}$ is called **closed**, and an element in the image of $d\colon A_{k-1} \to A_k$ is called **exact**. Thus, the $k$-th cohomology group measures closed elements in $A_k$ up to addition by exact ones.

The field of **homological algebra** provides a general framework for studying algebraic structures through sequences of abelian groups or vector spaces connected by differential operators, known as cochain or chain complexes. These complexes arise naturally across many areas of mathematics, including topology, geometry, and algebra, and their associated *cohomology groups* capture essential structural and invariance properties.

In this course, we focus only on one particular example: the so-called **de Rham cochain complex** in (18.4), where the vector spaces are the spaces of differential forms on a smooth manifold and the differential is given by the exterior derivative. The resulting cohomology groups are called the **de Rham cohomology groups** and are denoted by

$$H^k_{\mathrm{dR}}(M,\mathbb{R}) = \frac{\ker\big(d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)\big)}{\mathrm{Im}\,(d\colon \Omega^{k-1}(M) \longrightarrow \Omega^k(M))}$$

or simply $H^k(M,\mathbb{R})$.

**Example 18.6.** The 0-th cohomology group of any manifold is the vector space of locally-constant functions on $M$. Therefore, if $M$ is connected, then

$$H^0(M,\mathbb{R}) \cong \mathbb{R}.$$

**Exercise 18.7.** Find the degree one de Rham cohomology groups of $\mathbb{R}$ and $S^1$.

**Exercise 18.8.** Show that wedge product between differential forms descends to a product structure between de Rham cohomology classes making the total cohomology group $H^*(M) = \bigoplus_k H^k(M)$ a ring.

We will learn about a few results and techniques for calculating the cohomology groups of more complicated spaces in future sections.

When dealing with non-compact manifolds such as $\mathbb{R}^m$, it is often useful to restrict attention to differential forms with compact support. For every $k \geq 0$, let $\Omega^k_c(M) \subset \Omega^k(M)$ denote the subspace of differential $k$-forms with compact support; that is, every $\eta \in \Omega^k_c(M)$ vanishes outside a compact subset of $M$.

For each $k \geq 0$, the exterior derivative map

$$d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

restricts to a map

$$d\colon \Omega^k_c(M) \longrightarrow \Omega^{k+1}_c(M).$$

Therefore, we obtain a **compactly supported de Rham complex**

$$0 \longrightarrow \Omega^0_c(M) \xrightarrow{d} \Omega^1_c(M) \xrightarrow{d} \cdots \longrightarrow \Omega^{\dim M}_c(M) \xrightarrow{d} 0, \tag{18.5}$$

which gives rise to the compactly supported de Rham cohomology groups

$$H^k_{c,\mathrm{dR}}(M,\mathbb{R}) = \frac{\ker\left(d\colon \Omega^k_c(M) \longrightarrow \Omega^{k+1}_c(M)\right)}{\operatorname{Im}\left(d\colon \Omega^{k-1}_c(M) \longrightarrow \Omega^k_c(M)\right)},$$

often denoted more simply by $H^k_c(M,\mathbb{R})$. If $M$ is compact, then these groups coincide with the usual de Rham cohomology groups $H^k(M,\mathbb{R})$. For non-compact manifolds, however, the compactly supported cohomology groups differ in general and provide additional topological information.

**Exercise 18.9.** Find the compactly supported de Rham cohomology groups of $\mathbb{R}$.

## Solutions to exercises

**Exercise 18.7.** Since $\dim \mathbb{R} = \dim S^1 = 1$, every 1-form on $\mathbb{R}$ and $S^1$ is automatically closed. We need to find the subspace of exact 1-forms.

Starting with $\mathbb{R}$, every 1-form is a function multiple of $dx$ where $x$ is the global variable of $\mathbb{R}$. For every smooth 1-form $f(x)\,dx$, let

$$F(x) = \int_0^x f(t)\,dt.$$

By the Fundamental Theorem of Calculus, $dF = f\,dx$. Therefore, every 1-form is exact and $H^1(\mathbb{R}, \mathbb{R}) = 0$.

Thinking of $S^1$ as $\mathbb{R}/\mathbb{Z}$ where $\mathbb{Z}$ acts by translations by integers, since the 1-form $dx$ on $\mathbb{R}$ is invariant under the action of $\mathbb{Z}$, it descends to a nowhere vanishing 1-form on $S^1$. Furthermore, every 1-form on $S^1$ is of the form $f(x)\,dx$ for some function $f$ on $S^1$ which corresponds to a 1-periodic (i.e. $f(x+1) = f(x)$) function on $\mathbb{R}$. For $f(x)\,dx$ to be exact on $S^1$, i.e. $f(x)\,dx = dF(x)$ for some 1-periodic function $F$, we must have

$$0 = F(1) - F(0) = \int_0^1 f(t)\,dt.$$

Conversely, if $\int_0^1 f(t)\,dt = 0$, the function $F(x) = \int_0^x f(t)\,dt$ is 1-periodic and $dF = f(x)\,dx$. For every 1-form $f(x)\,dx$, we have

$$f(x)\,dx = a\,dx + (f(x) - a)\,dx,$$

where $a = \int_0^1 f(t)\,dt$ and $\int_0^1 (f(t) - a)\,dt = 0$. We conclude that $f(x)\,dx$ and $a\,dx$ have the same image in the quotient space

$$H^1_{\mathrm{dR}}(S^1, \mathbb{R}) = \frac{\Omega^1(S^1)}{\mathrm{Im}\,(d\colon \Omega^0(S^1) \longrightarrow \Omega^1(S^1))}.$$

Therefore, the class of $dx$ in $H^1_{\mathrm{dR}}(S^1, \mathbb{R})$ is a generator and

$$H^1_{\mathrm{dR}}(S^1, \mathbb{R}) \cong \mathbb{R}. \qquad \square$$

**Exercise 18.8.** To show that wedge product between differential forms descends to a product structure between de Rham cohomology classes, we must show that

- the wedge product of two closed forms is closed;
- the wedge product of a closed and an exact form is exact.

If $\eta_1$ and $\eta_2$ are two closed forms, then by Corollary 18.4.3, $\eta_1 \wedge \eta_2$ is closed as well.

Further, if $\eta_1 = d\vartheta$, then

$$d(\vartheta \wedge \eta_2) = d\vartheta \wedge \eta_2 \pm \vartheta \wedge d\eta_2 = d\vartheta \wedge \eta_2 = \eta_1 \wedge \eta_2,$$

proving the second property listed above. □

**Exercise 18.9.** The only compactly supported function $f$ satisfying $df = 0$ is the trivial constant function 0. Therefore,

$$H^0_{c,\mathrm{dR}}(\mathbb{R}, \mathbb{R}) = \ker\big(d\colon \Omega^0_c(\mathbb{R}) \longrightarrow \Omega^1_c(\mathbb{R})\big) = 0.$$

If a compactly supported 1-form $f(x)\,dx$ is of the form $dF$ for some compactly supported function $F$, then

$$\int_{-\infty}^{\infty} f(t)\,dt = F(+\infty) - F(-\infty) = 0.$$

Conversely, if $\int_{-\infty}^{\infty} f(t)\,dt = 0$, then the function

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

is compactly supported and satisfies $dF = f(x)\,dx$. Therefore, the $\mathbb{R}$-linear map

$$\int\colon \Omega^1_c(\mathbb{R}) \longrightarrow \mathbb{R}, \qquad f(x)\,dx \longmapsto \int_{-\infty}^{\infty} f(x)\,dx$$

descends to an isomorphism

$$\int\colon H^1_{c,\mathrm{dR}}(\mathbb{R}, \mathbb{R}) = \frac{\Omega^1_c(\mathbb{R})}{\mathrm{Im}\,(d\colon \Omega^0_c(\mathbb{R}) \longrightarrow \Omega^1_c(\mathbb{R}))} \xrightarrow{\cong} \mathbb{R}.$$

□

Chapter 19

# Curl, Divergence, and $d \circ d = 0$

Two fundamental identities in 3-dimensional vector calculus are

$$\nabla \times (\nabla f) = 0 \quad \text{and} \quad \nabla \cdot (\nabla \times \vec{X}) = 0. \tag{19.1}$$

The first identity states that the **curl of a gradient** is always zero, meaning the gradient of a scalar field is irrotational. The second states that the **divergence of a curl** is always zero, implying that the curl of a vector field is divergence-free. These identities are consequences of the symmetry of second derivatives and form the backbone of many theoretical results in vector calculus. In physics, they are deeply tied to the structure of Maxwell's equations: for example, the identity $\nabla \cdot (\nabla \times \vec{X}) = 0$ ensures the absence of magnetic monopoles in classical electromagnetism. Similarly, the irrotational nature of conservative force fields, such as gravitational or electrostatic fields, follows from $\nabla \times (\nabla f) = 0$. These properties are also central in the formulation of potential theory and in the analysis of fluid flow and electromagnetic fields.

In this lecture, we show that these results are equivalent to the identity $d \circ d$ on differential forms.

**Definition 19.1.** For a smooth function $f\colon V \subset \mathbb{R}^3 \to \mathbb{R}$, the gradient vector field of $f$ is

$$\nabla f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\, \partial_{x_i}.$$

This is a special case of (16.2) where the standard metric on $\mathbb{R}^3$ is used. For a vector field

$$X = \sum_{i=1}^{3} a_i(x)\,\partial_{x_i}$$

on $V \subset \mathbb{R}^3$, the **curl** of $X$, denoted by $\nabla \times X$, is the vector field

$$\nabla \times X = \det \begin{bmatrix} \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ a_1 & a_2 & a_3 \end{bmatrix},$$

which expands to

$$\nabla \times X = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\partial_{x_1} + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\partial_{x_2} + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\partial_{x_3}.$$

Lastly, the divergence of a vector field $X$ is the scalar function

$$\nabla \cdot X = \sum_{i=1}^{3} \frac{\partial a_i(x)}{\partial x_i}.$$

The divergence of a vector field admits a generalization to any manifold equipped with a volume form, which we will encounter later in this lecture. In contrast, the notion of curl is intrinsically three-dimensional.

**Theorem 19.2.** *For an open subset $V \subset \mathbb{R}^3$, let $\mathrm{Vect}(V)$ denote the space of smooth vector fields on $V$. Then the following diagram commutes:*

$$\begin{tikzcd} C^\infty(V,\mathbb{R}) \arrow[r, "\nabla"] \arrow[d, "\mathrm{id}"] & \mathrm{Vect}(V) \arrow[r, "\nabla\times"] \arrow[d, "(16.2)"] & \mathrm{Vect}(V) \arrow[r, "\nabla\cdot"] \arrow[d, "\iota_{(-)}\omega_{\mathrm{std}}"] & C^\infty(V,\mathbb{R}) \arrow[d, "\cdot\,\omega_{\mathrm{std}}"] \\ \Omega^0(V) \arrow[r, "d"] & \Omega^1(V) \arrow[r, "d"] & \Omega^2(V) \arrow[r, "d"] & \Omega^3(V) \end{tikzcd}$$

*Here, the first column is the identity map between smooth functions; the second column represents the identification in* (16.2) *between vector fields and 1-forms using the standard metric on $\mathbb{R}^3$; the third column is an isomorphism mapping a vector field $X$ to the 2-form*

$$\iota_X \omega_{\mathrm{std}} = \iota_X(dx_1 \wedge dx_2 \wedge dx_3);$$

*and the last column identifies functions and* 3*-forms by mapping a function $f$ to the 3-form $f\,\omega_{\mathrm{std}} = f\,dx_1 \wedge dx_2 \wedge dx_3$. In other words, there is a dictionary (i.e. identification of vector spaces) between the top and bottom rows such that the vector calculus identities in* (19.1) *correspond to the differential form identity $d \circ d = 0$.*

**Proof.** The proof is purely computational, as all the maps have explicit formulas. We go over each square for the sake of completeness.

In the first square, using the standard metric from (16.2), the gradient $\nabla f$ corresponds to the differential

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\, dx_i.$$

The second square commutes because

$$\begin{aligned}\iota_{\nabla\times X}\omega_{\mathrm{std}} &= \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) dx_2 \wedge dx_3 \\ &\quad + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 \wedge dx_2\end{aligned}$$

is equal to

$$\begin{aligned}d(a_1\, dx_1 + a_2\, dx_2 + a_3\, dx_3) &= -\frac{\partial a_1}{\partial x_2}\, dx_1 \wedge dx_2 - \frac{\partial a_1}{\partial x_3}\, dx_1 \wedge dx_3 \\ &\quad + \frac{\partial a_2}{\partial x_1}\, dx_1 \wedge dx_2 - \frac{\partial a_2}{\partial x_3}\, dx_2 \wedge dx_3 \\ &\quad + \frac{\partial a_3}{\partial x_1}\, dx_1 \wedge dx_3 + \frac{\partial a_3}{\partial x_2}\, dx_2 \wedge dx_3 \\ &= \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) dx_2 \wedge dx_3 \\ &\quad + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) dx_3 \wedge dx_1 \\ &\quad + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 \wedge dx_2.\end{aligned}$$

Finally, the last square commutes because

$$\begin{aligned}d\left(\iota_{(\sum a_i\, \partial_{x_i})}\omega_{\mathrm{std}}\right) &= d\left(a_1\, dx_2 \wedge dx_3 + a_2\, dx_3 \wedge dx_1 + a_3\, dx_1 \wedge dx_2\right) \\ &= \left(\sum_{i=1}^{3} \frac{\partial a_i(x)}{\partial x_i}\right) dx_1 \wedge dx_2 \wedge dx_3.\end{aligned}$$

□

**Definition 19.3.** Given a smooth manifold $M$ with a volume form $\omega$, the **divergence** of a smooth vector field $X$ with respect to $\omega$ is the unique smooth function $f = \mathrm{Div}_\omega(X)$ such that

$$d(\iota_X\omega) = f\omega.$$

Note that the interior product $\iota_X$ decreases the degree of a differential form by 1, and the exterior derivative $d$ increases it by 1, returning to the degree of $\omega$. Since every top-degree form on $M$ is a scalar multiple of the volume form $\omega$, the function $f$ is well-defined.

**Definition 19.4.** Given a smooth manifold $M$ with a metric $g$, let $\omega_g$ denote the canonical volume form associated to $g$. For every smooth function $f\colon M \to \mathbb{R}$, the **Laplacian** of $f$, denoted by $\Delta f$, is the function

$$\Delta f := \mathrm{Div}_{\omega_g}(\nabla f),$$

where $\nabla f$ is the gradient vector field associated to $f$ as in (16.2).

**Exercise 19.5.** Find an explicit formula for $\Delta f$ in terms of the partial derivatives of $f$ and the components $g_{ij}(x)$ of the metric in an arbitrary local coordinate chart $x = (x_1, \dots, x_m)$. Also, if a vector field $X$ has local equation $X = \sum_i a_i(x)\, \partial_{x_i}$, write an explicit equation for $\mathrm{Div}_{\omega_g}(X)$ in terms of $g_{ij}$ and partial derivatives of $a_i$.

**Exercise 19.6.** Consider the upper half plane $\mathcal{H}$ with the Poincare metric $g = \frac{dx^2+dy^2}{y^2}$. Let

$$X = (1 + x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}.$$

Show that

$$\mathrm{Div}_{\omega_g}(X) = 0.$$

Is $X$ gradient of a function?

## Solutions to exercises

**Exercise 19.5.** First, we find an explicit formula for $\mathrm{Div}_{\omega_g}(X)$.

Following the definition, we have

$$d\left(\iota_X \sqrt{\det[g_{kl}(x)]}\, dx_1 \wedge \cdots \wedge dx_m\right) =$$

$$d\left(\sum_{i=1}^{m} (-1)^{i-1} a_i \sqrt{\det[g_{kl}(x)]}\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \cdots \wedge dx_m\right) =$$

$$\sum_{i=1}^{m} \frac{\partial\left(a_i \sqrt{\det[g_{kl}(x)]}\right)}{\partial x_i}\; dx_1 \wedge \cdots \wedge dx_m =$$

$$\left(\sum_{i=1}^{m} \frac{\partial a_i}{\partial x_i}\right)\omega_g + \frac{1}{2}\left(\sum_{i=1}^{m} a_i \frac{\partial \log \det[g_{kl}]}{\partial x_i}\right)\omega_g.$$

Therefore,

$$\mathrm{Div}_{\omega_g}(X) = \sum_{i=1}^{m} \frac{\partial a_i}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{m} a_i \frac{\partial \log \det[g_{kl}]}{\partial x_i}.$$

Next, in order to compute $\Delta f$, we apply the formula above to

$$X = \nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_j} \partial_{x_i}.$$

Since

$$a_i = \sum_j g^{ij} \frac{\partial f}{\partial x_j},$$

we get

$$\Delta f = \sum_{i,j} g^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i,j} \frac{\partial g^{ij}}{\partial x_i}\frac{\partial f}{\partial x_j} + \frac{1}{2}\sum_{i,j} g^{ij} \frac{\partial f}{\partial x_j}\frac{\partial \log \det[g_{kl}]}{\partial x_i}.$$

If we directly use the formula for Laplacian we get the more compact formula:

$$\Delta f = \frac{1}{\sqrt{\det[g_{kl}(x)]}} \sum_{i,j} \frac{\partial}{\partial x_i}\left(\sqrt{\det[g_{kl}(x)]}\, g^{ij} \frac{\partial f}{\partial x_j}\right).$$

**Remark 19.7.** Note that the Laplacian is a second-order differential operator whose principal (second-order) part is

$$\sum_{i,j} g^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j},$$

corresponding to the action of the inverse metric on the Hessian of $f$. The remaining lower-order terms account for the variation of the metric tensor and the volume form.

$\square$

**Exercise 19.6.** The volume form of $g$ with respect to the counter clock wise orientation is

$$\omega_g = y^{-2} dx \wedge dy$$

Therefore

$$\iota_X \omega_g = (y^{-2} + x^2 y^{-2} - 1) dy - 2xy^{-1} dx,$$

and

$$d\iota_X \omega_g = \left(2xy^{-2} - 2xy^{-2}\right) dx \wedge dy = 0$$

For $X$ to be the gradient of a function $f$, we must have

$$\nabla f = y^2 \left( \frac{\partial f}{\partial x} \partial_x + \frac{\partial f}{\partial y} \partial_y \right) = (1 + x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y},$$

or equivalently,

$$\frac{\partial f}{\partial x} = y^{-2}(1 + x^2 - y^2), \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy^{-1}. \tag{19.2}$$

Integrating the second equation with respect to $y$ gives

$$f(x, y) = 2x \ln(y) + g(x).$$

Differentiating this identity with respect to $x$ yields

$$\frac{\partial f}{\partial x} = 2 \ln(y) + g'(x).$$

Equating with the earlier expression for $\frac{\partial f}{\partial x}$ gives

$$2 \ln(y) + g'(x) = y^{-2}(1 + x^2 - y^2),$$

or

$$g'(x) = y^{-2}(1 + x^2 - y^2) - 2 \ln(y).$$

This is a contradiction, since the left-hand side depends only on $x$, while the right-hand side depends on both $x$ and $y$.

One can also obtain a contradiction by differentiating the first equation in (19.2) with respect to $y$, and the second one with respect to $x$, and observing that the right-hand sides do not agree.

$\square$

Chapter 20

# Integration and Stokes' Theorem

In multivariable calculus, we learn to integrate functions over regions in $\mathbb{R}^m$. In this lecture, we take a new point of view: we interpret a multivariable integral

$$\int_V f(x_1, \dots, x_m)\, dx_1 \cdots dx_m$$

as the integral of an $m$-form, namely $f(x_1, \dots, x_m)\, dx_1 \wedge \cdots \wedge dx_m$, over the region $V \subset \mathbb{R}^m$ or $V \subset \mathbb{H}_m$. This perspective not only clarifies the geometric meaning of the integrand, but also extends naturally to general manifolds, once orientation is properly accounted for. In fact, the familiar change of variables formula involving the Jacobian determinant fits seamlessly into this framework and shows that integration of top-degree forms on oriented manifolds is well-defined.

**Remark 20.1.** Since manifolds with boundary play an important role in integration on manifolds, we will be more precise in this section and take charts to have image in $\mathbb{H}_m$ to account for the possibility of boundary.

**Theorem 20.2.** *Suppose $M$ is an oriented smooth $m$-manifold. There exists an $\mathbb{R}$-linear map*

$$\int_M : \Omega^m_c(M) \longrightarrow \mathbb{R}$$

*with the following property: if $\varphi\colon U \longrightarrow V \subset \mathbb{R}^m$ is a chart compatible with the orientation, and $\eta = f(x)\, dx_1 \wedge \cdots \wedge dx_m$ is an $m$-form compactly* *supported in $U$, then*

$$\int_M \eta = \int_{\mathbb{R}^m} f(x_1, \dots, x_m)\, dx_1 \cdots dx_m$$

*in the sense of multivariable calculus.*

**Proof.** Fix $\eta \in \Omega_c^m(M)$ and let $K$ be the (compact) support of $\eta$. Choose a finite collection of charts

$$\mathcal{C} = \{\varphi_i \colon U_i \longrightarrow V_i \subset \mathbb{H}_m\}_{1 \le i \le \ell},$$

compatible with the orientation, such that

$$K \subset \bigcup_{i=1}^{\ell} U_i.$$

If $M \neq K$, let $U_0 \subset M$ be the complement of $K$. Let $\{\varrho_i \colon U_i \longrightarrow [0,1]\}_{i=0}^{\ell}$ be a partition of unity subordinate to the open cover $\{U_i\}_{i=0}^{\ell}$, and set $\eta_i = \varrho_i \eta$. Note that $\eta_0 = 0$ and $\eta = \sum_{i=1}^{\ell} \eta_i$. Since $\eta_i$ is an $m$-form supported in $U_i$, it has an expression

$$\eta_i = f_i\, dx_1 \wedge \cdots \wedge dx_m$$

for some smooth function $f_i$ compactly supported in $V_i$, and we define

$$I_i = \int_{\mathbb{H}_m} f_i(x_1, \dots, x_m)\, dx_1 \cdots dx_m$$

in the sense of multivariable calculus. Finally, we define

$$\int_M \eta := \sum_{i=1}^{\ell} I_i.$$

We need to show that the latter is independent of the choices made.

Suppose

$$\mathcal{C}' = \{\varphi'_j \colon U'_j \longrightarrow V'_j \subset \mathbb{H}_m\}_{1 \le j \le k}$$

is another such collection of charts, and let $\{\varrho'_j \colon U'_j \longrightarrow [0,1]\}_{j=0}^{k}$ be a partition of unity subordinate to $\{U'_j\}_{j=0}^{k}$, where $U'_0 = U_0$. Then the double-indexed finite collections

$$\mathcal{C}_1 = \{\varphi_{ij} = \varphi_i|_{U_i \cap U'_j} \colon U_i \cap U'_j \longrightarrow \mathbb{H}_m\}_{1 \le i \le \ell,\ 1 \le j \le k}$$

and

$$\mathcal{C}_2 = \{\varphi'_{ij} = \varphi'_j|_{U_i \cap U'_j} \colon U_i \cap U'_j \longrightarrow \mathbb{H}_m\}_{1 \le i \le \ell,\ 1 \le j \le k}$$

both cover $K$ and refine $\mathcal{C}$ and $\mathcal{C}'$, respectively. Furthermore,

$$\big\{\varrho_{ij} = \varrho_i \cdot \varrho'_j \colon U_i \cap U'_j \longrightarrow [0,1]\big\}_{1 \le i \le \ell,\ 1 \le j \le k} \cup \{\varrho_{00} \colon U_0 \longrightarrow [0,1]\},$$

where

$$\varrho_{00} = \varrho_0 + \varrho'_0 - \varrho_0 \varrho'_0,$$

is a partition of unity subordinate to the cover

$$\{U_i \cap U'_j\}_{1\le i\le \ell,\ 1\le j\le k} \cup \{U_0\}.$$

For $1 \le i \le \ell$, $1 \le j \le k$, let $I_{ij}$ and $I'_{ij}$ denote the integrals of $\eta_{ij} = \varrho_{ij}\eta$ with respect to the chart maps $\varphi_{ij}$ and $\varphi'_{ij}$, respectively.

We claim that

$$\sum_{i=1}^{\ell} I_i = \sum_{1\le i\le \ell,\ 1\le j\le k} I_{ij} = \sum_{1\le i\le \ell,\ 1\le j\le k} I'_{ij} = \sum_{j=1}^{k} I'_j, \tag{20.1}$$

where $I'_j$ denotes the integral of $\eta'_j = \varrho'_j\eta$ with respect to the chart map $\varphi'_j$.

The first and last equalities follow from the additivity of integration over $\mathbb{H}_m$, which gives

$$I_i = \sum_{1\le j\le k} I_{ij}, \qquad I'_j = \sum_{1\le i\le \ell} I'_{ij}.$$

It remains to show that $I_{ij} = I'_{ij}$ for all $1 \le i \le \ell$, $1 \le j \le k$. Both $I_{ij}$ and $I'_{ij}$ are the integrals of the same $m$-form $\eta_{ij}$ with respect to two (potentially different) chart maps $\varphi_{ij}$ and $\varphi'_{ij}$. Suppose

$$\eta_{ij} = f(x)\, dx_1 \wedge \cdots \wedge dx_m$$

in coordinates via $\varphi_{ij}$ and

$$\eta_{ij} = g(x)\, dx_1 \wedge \cdots \wedge dx_m$$

in coordinates via $\varphi'_{ij}$. If $\psi(x) = \varphi'_{ij} \circ \varphi_{ij}^{-1}(x)$ is the transition map, then

$$f(x)\, dx_1 \wedge \cdots \wedge dx_m = \psi^*(g\, dx_1 \wedge \cdots \wedge dx_m) = \det(d\psi)\, g(\psi(x))\, dx_1 \wedge \cdots \wedge dx_m.$$

Since both charts are compatible with the orientation, we have $\det(d\psi) > 0$. It follows from the **change of variables formula** in multivariable calculus that

$$\begin{aligned} I'_{ij} &= \int_{\mathbb{H}_m} g(x)\, dx_1 \cdots dx_m \\ &= \int_{\mathbb{H}_m} \det(d\psi)\, g(\psi(x))\, dx_1 \cdots dx_m \\ &= \int_{\mathbb{H}_m} f(x)\, dx_1 \cdots dx_m = I_{ij}. \end{aligned}$$

□

**Exercise 20.3.** Find the integral $\int_{S^2} \omega_2$ where $\omega_2$ is the standard volume (area) form of $S^2$ as expressed in Exercise 17.10.

Next, we learn about Stokes' Theorem that expresses a deep relationship between differentiation and integration, generalizing the Fundamental Theorem of Calculus (FTC) to higher dimensions and to integration over manifolds. Just as FTC relates the integral of a derivative over an interval to the values of a function at the boundary points, Stokes' Theorem relates the integral of an exact differential form over a manifold to the integral of the differential form itself over the boundary of that manifold.

**Theorem 20.4.** *(Stokes' Theorem) Suppose $M$ is a smooth oriented $m$-manifold (possibly) with boundary $\partial M$ and $\eta$ is a compactly supported $(m-1)$-form on $M$. Then*

$$\int_M d\eta = \int_{\partial M} \eta,$$

*where the induced orientation on $\partial M$ is chosen such that, for an outward-pointing vector field $\vec{n}$ along $\partial M$, the vector bundle isomorphism*

$$TM|_{\partial M} = \mathbb{R} \cdot \vec{n} \oplus T\partial M$$

*is orientation preserving (see solution to Exercise 15.19).*

Many classical results in calculus are special cases of the general Stokes' Theorem above. For instance, in addition to the Fundamental Theorem of Calculus (FTC), we have the following:

- **Green's Theorem (in the plane):**
$$\oint_{\partial R} P\,dx + Q\,dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$
This is a special case of Stokes' Theorem in dimension 2, where $\oint$ denotes integration over $\partial R$ with respect to the counterclockwise orientation.
- **Curl Theorem:**
$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$
This is the general theorem applied to a 2-dimensional surface $S \subset \mathbb{R}^3$ with boundary. Under the dictionary of Theorem 19.2, this corresponds to Theorem 20.4 applied to a 1-form.
- **Divergence Theorem (Gauss' Theorem):**
$$\iiint_V (\nabla \cdot \vec{F})\,dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$$
This corresponds to Stokes' Theorem on a 3-dimensional domain $V$ with boundary surface $\partial V$. Under the dictionary of Theorem 19.2, this corresponds to Theorem 20.4 applied to a 2-form. More generally, for any $m$-manifold $M$ with a volume form $\omega$ and boundary

$\partial M$, and with divergence defined as in (19.3), the **Divergence Theorem** reads:

$$\int_M \mathrm{Div}_\omega(X)\,\omega = \int_{\partial M} \iota_X \omega.$$

If $\omega = \omega_g$ is the volume form associated to a Riemannian metric, $\vec{n}$ is the orthonormal outward unit vector field along $\partial M$, and $\omega_{\partial M}$ is the volume form of $\partial M$ (with respect to the induced metric), we can re-write the identity above as

$$\int_M \mathrm{Div}_\omega(X)\,\omega = \int_{\partial M} X \cdot \vec{n}\;\omega_{\partial M},$$

where $X \cdot \vec{n} = g(X, \vec{n})$ measures the flow of $X$ across the boundary of $M$.

- **Cauchy Integral Formula (Complex Analysis):**

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\,dz$$

This follows from Stokes' Theorem applied to the closed 1-form $\omega = \frac{f(z)}{z-z_0} dz$ on a cylindrical domain in $\mathbb{C}$, using the fact that holomorphic functions satisfy $d\omega = 0$.

**Proof of Stokes' Theorem.** As we observed in the proof of Theorem 20.2, any integral can be written as a finite (or countable) sum of integrals over charts. Thus, it suffices to prove the result for a compactly supported differential $(m{-}1)$-form

$$\eta = \sum_{i=1}^{m} a_i(x)\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_m$$

on $\mathbb{H}_m = \mathbb{R}_{\geq 0} \times \mathbb{R}^{m-1}$.

We compute

$$d\eta = \sum_{i=1}^{m} (-1)^{i-1} \frac{\partial a_i(x)}{\partial x_i}\,dx_1 \wedge \cdots \wedge dx_m.$$

For $i > 1$, integrating first with respect to the $i$-th variable we get

$$\int_{\mathbb{H}_m} \frac{\partial a_i(x)}{\partial x_i}\,dx_1 \wedge \cdots \wedge dx_m = \int_{\mathbb{H}_{m-1}} \left( \int_{-\infty}^{\infty} \frac{\partial a_i(x)}{\partial x_i}\,dx_i \right) dx_1 \cdots \widehat{dx_i} \cdots dx_m = 0,$$

since by the Fundamental Theorem of Calculus,

$$\int_{-\infty}^{\infty} \frac{\partial a_i}{\partial x_i}\,dx_i = 0.$$

For $i=1$, integrating first with respect to $x_1$ and using the Fundamental Theorem of Calculus gives:

$$\begin{aligned}\int_{\mathbb{H}_m} \frac{\partial a_1}{\partial x_1}\, dx_1 \wedge \cdots \wedge dx_m &= \int_{\mathbb{R}^{m-1}} \left( \int_0^\infty \frac{\partial a_1(x_1, x_2, \ldots, x_m)}{\partial x_1}\, dx_1 \right) dx_2 \cdots dx_m \\ &= \int_{\mathbb{R}^{m-1}} -a_1(0, x_2, \ldots, x_m)\, dx_2 \cdots dx_m.\end{aligned}$$

On the other hand, the outward unit normal vector field to $\mathbb{H}_m$ is $-\partial_{x_1}$. Therefore, the coordinates $(x_2, \ldots, x_m)$ on $\partial\mathbb{H}_m$ induce the opposite orientation from the one inherited from $M$. We conclude that

$$-\int_{\mathbb{R}^{m-1}} a_1(0, x_2, \ldots, x_m)\, dx_2 \cdots dx_m = \int_{\partial M} \eta|_{\partial M}.$$

$\square$

**Exercise 20.5.** Redo Exercise 20.3 using the presentation of $\omega_2$ Example 17.9 and Stokes' Theorem.

**Exercise 20.6.** Suppose $\gamma : S^1 \to \mathbb{R}^2$ is a smooth embedding. Compute the integral $\int_{S^1} \gamma^*\theta$ when

$$\theta = xy^2 dx + x^2 y dy$$

**Exercise 20.7.** Show that the 1-form

$$\eta = \frac{xdy - ydx}{x^2 + y^2}$$

on $\mathbb{R}^2 - \{0\}$ is closed but not exact. Calculate the integral $\int_C \eta$ on the ellipse

$$C = \{(x, y) \in \mathbb{R}^2 \colon x^2 + 2y^2 = 1\}.$$

**Exercise 20.8.** Let $\eta$ be the 2-form on $\mathbb{R}^3 - \{0\}$ defined by

$$\eta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

Let $\Sigma \subset \mathbb{R}^3 - \{0\}$ be a smooth compact surface that is the boundary $\partial U$ of a compact 3-manifold-with-boundary $U \subset \mathbb{R}^3$. Let's agree to give the "bounded domain" $U$ the orientation it inherits from $\mathbb{R}^3$, and then use this to induce the corresponding "out-pointing" boundary orientation on $\Sigma = \partial U$. Prove that

$$\frac{1}{4\pi} \int_\Sigma \eta = \begin{cases} 1 & \text{if} \quad 0 \in U, \\ 0 & \quad \text{otherwise.} \end{cases}$$

**Exercise 20.9.** In Exercise 8.14, we showed that for $a > b > 0$, the surface

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid (r - a)^2 + z^2 = b^2\}$$

is a diffeomorphic to a 2-torus. Here, $r^2 = x^2 + y^2$. Find the area of $M$ with respect to the standard metric on $\mathbb{R}^3$.

## Solutions to exercises

**Exercise 20.3.** We compute $\int_{S^2} \omega_2$ in two ways.

**Remark 20.10.** For any proper open subset $U \subsetneq S^2$, the restriction $\omega_2|_U$ does not have compact support, but $\int_U \omega_2$ is still well-defined and can be computed using any system of local coordinates. We have used differential forms with compact support primarily to ensure the finiteness of integrals. Nevertheless, since $\int_{S^2} \omega_2$ is finite, integrating $\omega_2$ over any chart yields a finite value. Moreover, if the complement of $U$ has measure zero, then

$$\int_{S^2} \omega_2 = \int_U \omega_2.$$

Thus, in many cases, integrating over a single chart suffices to compute the integral over the entire manifold.

First, we calculate the area of the upper hemisphere, which is half of the total area. To do this, by Example 17.10, we need to integrate

$$\frac{1}{x_0}\, dx_1 \wedge dx_2$$

over the upper hemisphere, which is the graph of

$$\varphi\colon B_1 \to \mathbb{R}^3, \qquad (x_1, x_2) \mapsto \left( x_0 = \sqrt{1 - x_1^2 - x_2^2},\, x_1,\, x_2 \right).$$

Therefore,

$$\frac{1}{2}\text{area of } S^2 = \int_{\mathrm{Image}(\varphi)} \frac{1}{x_0}\, dx_1 \wedge dx_2 = \int_{B_1} \varphi^* \left( \frac{1}{x_0}\, dx_1 \wedge dx_2 \right).$$

Changing to polar coordinates $(r, \theta)$ on the disk of radius one $B_1 \subset \mathbb{R}^2$, the integral becomes

$$\int_{r=0}^{1} \int_{\theta=0}^{2\pi} \frac{1}{\sqrt{1-r^2}}\, r\, dr\, d\theta = 2\pi \int_0^1 \frac{r}{\sqrt{1-r^2}}\, dr = -2\pi \sqrt{1-r^2}\Big|_0^1 = 2\pi.$$

Therefore, $\int_{S^2} \omega_2 = 4\pi$.

In the second approach, we cover all but one point of $S^2$ using a single chart – namely, the stereographic projection map

$$\varphi_+ \colon U_+ \to \mathbb{R}^2$$

from (2.3). Since $U_+$ is dense in $S^2$, we have

$$\int_{S^2} \omega_2 = \int_{\mathbb{R}^2} (\varphi_+^{-1})^* \omega_2.$$

We compute:

$$\varphi_+^{-1}(x_1, x_2) = \frac{1}{1+|x|^2}\left(|x|^2 - 1,\ 2x_1,\ 2x_2\right).$$

Using polar coordinates $(r, \theta)$ on $\mathbb{R}^2$ and cylindrical coordinates $(z = x_0, R, \vartheta)$ on $\mathbb{R}^3$, this becomes

$$(r, \theta) \mapsto (z, R, \vartheta) = \left(\frac{r^2-1}{r^2+1},\ \frac{2r}{r^2+1},\ \theta\right).$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^2} (\varphi_+^{-1})^* \omega_2 &= \int_{\mathbb{R}^2} (\varphi_+^{-1})^* \left(\frac{1}{z} R\, dR \wedge d\vartheta\right) \\
&= \int_{\mathbb{R}^2} \frac{r^2+1}{r^2-1} \cdot \frac{2r}{r^2+1}\, d\left(\frac{2r}{r^2+1}\right) \wedge d\theta \\
&= \int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} \frac{-4r}{(r^2+1)^2}\, dr \wedge d\theta \\
&= 2\pi \int_0^{\infty} \frac{-4r}{(r^2+1)^2}\, dr = -4\pi \left[\frac{1}{r^2+1}\right]_0^{\infty} = 4\pi.
\end{aligned}$$

□

**Exercise 20.5.** In Example 17.9, the form $\omega_2$ is the restriction of

$$x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

to $S^2$. By Stokes' Theorem,

$$\int_{B_1} d(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy) = \int_{S^2} x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy,$$

where $B_1$ is the unit ball in $\mathbb{R}^3$ whose boundary is $S^2$. Since

$$d\big(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy\big) = 3\, dx \wedge dy \wedge dz,$$

we obtain

$$\text{area of } S^2 = 3 \int_{B_1} dx \wedge dy \wedge dz = 3 \times \text{volume of } B_1 = 3 \cdot \frac{4}{3}\pi = 4\pi.$$

Here, the volume of the unit ball can easily be computed using spherical coordinates in $\mathbb{R}^3$. □

**Exercise 20.6.** By Jordan Curve Theorem [**Kur66**], the image $C \subset \mathbb{R}^2$ of $\gamma$ divides the plane into exactly two connected components: a bounded interior $R$ and an unbounded exterior, with $C$ as their common boundary.

We have

$$d\theta = \frac{\partial(xy^2)}{\partial y} dy \wedge dx + \frac{\partial(x^2y)}{\partial x} dx \wedge dy = 2xy\, dy \wedge dx + 2xy\, dx \wedge dy = 0$$

Therefore, by Stokes' Theorem,

$$\int_{S^1} \gamma^*\theta = \int_C \theta = \int_R d\theta = 0.$$

**Exercise 20.7.** We have

$$\begin{aligned} d\eta &= \frac{\partial\left(\frac{x}{x^2+y^2}\right)}{\partial x}\, dx \wedge dy + \frac{\partial\left(\frac{y}{x^2+y^2}\right)}{\partial y}\, dx \wedge dy \\ &= \frac{y^2 - x^2}{(x^2+y^2)^2}\, dx \wedge dy + \frac{x^2 - y^2}{(x^2+y^2)^2}\, dx \wedge dy = 0. \end{aligned}$$

Writing $\eta$ in polar coordinates gives a simpler proof of closedness and an easier calculation of $\int_C \eta$. Since

$$\theta = \tan^{-1}\left(\frac{y}{x}\right),$$

we get

$$d\theta = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{\partial\theta}{\partial x}\, dx + \frac{\partial\theta}{\partial y}\, dy.$$

By the chain rule:

$$\begin{aligned} \frac{\partial\theta}{\partial x} &= \frac{d}{dx}\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}, \\ \frac{\partial\theta}{\partial y} &= \frac{d}{dy}\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}. \end{aligned}$$

Therefore,

$$d\theta = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy = \frac{x\, dy - y\, dx}{x^2+y^2} = \eta.$$

Let $S^1_\epsilon$ denote the circle of radius $\epsilon$ centered at the origin. For $\epsilon > 0$ sufficiently small, $S^1_\epsilon$ and $C$ bound an annular region $R$. By Stokes' Theorem, and noting that the orientation of $S^1_\epsilon$ is opposite to that of $C$ (due to outward normal vectors pointing in opposite directions), we have

$$0 = \int_R d\eta = \int_C \eta - \int_{S^1_\epsilon} \eta.$$

Therefore,

$$\int_C \eta = \int_{S^1_\epsilon} \eta = \int_{S^1_\epsilon} d\theta = 2\pi,$$

which simply measures the total change in angle along $C$.

Since the integral is nonzero, $\eta$ is not exact (this can also be seen from the fact that $\theta$ is multivalued). If $\eta$ were exact, Stokes' Theorem would have implied that $\int_C \eta = 0$. $\square$

**Exercise 20.8.** The 2-form $\eta$ is defined outside the origin and satisfies (check for yourself)

$$d\eta = 0.$$

If $U$ does not include the origin, the result follows from Stokes' theorem. If $U$ includes the origin, then for $\varepsilon > 0$ sufficiently small, $U$ contains the closure of the open ball

$$B_\varepsilon = \{x \in \mathbb{R}^3 \colon |x| < \varepsilon\}.$$

Let $V = U \setminus B_\varepsilon(0)$. Then $\partial V$ consists of $\Sigma = \partial U$ and the 2-sphere $S^2_\varepsilon$ of radius $\varepsilon$. By Stokes' theorem and the orientation convention (as in the previous exercise), we have

$$\int_\Sigma \eta = \int_{S^2_\varepsilon} \eta.$$

Restricted to $S^2_\varepsilon$, we have

$$\eta|_{S^2_\varepsilon} = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{\varepsilon^3}.$$

Therefore,

$$\int_{S^2_\varepsilon} \eta = \varepsilon^{-3} \int_{S^2_\varepsilon} x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

By Stokes' theorem again,

$$\begin{aligned}\int_{S^2_\varepsilon} x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy &= \int_{\overline{B_\varepsilon(0)}} d(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy) \\ &= \int_{\overline{B_\varepsilon(0)}} 3\, dx \wedge dy \wedge dz = 3 \operatorname{vol}(\overline{B_\varepsilon(0)}) = 4\pi\varepsilon^3.\end{aligned}$$

Putting everything together, we obtain

$$\int_\Sigma \eta = \varepsilon^{-3}(4\pi\varepsilon^3) = 4\pi.$$

$\square$

**Exercise 20.9.** With respect to the cylindrical coordinates $(r, \vartheta, z)$ on $\mathbb{R}^3$, we have the identification $S^1 \times S^1 \to M$ given by

$$h \colon S^1 \times S^1 \to \mathbb{R}^3, \quad (\theta, \varphi) \mapsto (a + b\cos(\varphi), \theta, b\sin(\varphi)),$$

where $\theta$ and $\varphi$ are the angular variables on $S^1 \subset \mathbb{C}$.

The manifold $M$ can also be seen as a (regular) level set of the function

$$f\colon \mathbb{R}^3 \to \mathbb{R}, \quad (r, \vartheta, z) \mapsto (r-a)^2 + z^2.$$

With respect to the Euclidean coordinates $(x, y, z)$ and the standard metric on $\mathbb{R}^3$, the gradient vector field of $f$ is

$$\nabla f = 2\left\{ \frac{(r-a)x}{r} \frac{\partial}{\partial x} + \frac{(r-a)y}{r} \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right\}.$$

To find the area form of $M$, we need the normal vector field

$$n = \frac{\nabla f}{|\nabla f|}$$

along $M$. We have

$$|\nabla f|_M = 2\sqrt{(r-a)^2 + z^2} = 2b.$$

Therefore,

$$n = \frac{1}{b}\left\{ \frac{(r-a)x}{r} \frac{\partial}{\partial x} + \frac{(r-a)y}{r} \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right\},$$

and the area form $\omega$ of $M$ is

$$\omega = \iota_n (dx \wedge dy \wedge dz).$$

It is easier to write everything in cylindrical coordinates. In $(r, \vartheta, z)$ coordinates,

$$dx \wedge dy \wedge dz = r\, dr \wedge d\vartheta \wedge dz$$

and

$$n = \frac{1}{b}\left\{ (r-a) \frac{\partial}{\partial r} + z \frac{\partial}{\partial z} \right\}.$$

Therefore,

$$\omega = \frac{zr}{b}\, dr \wedge d\vartheta + \frac{r(r-a)}{b}\, d\vartheta \wedge dz.$$

We conclude that

$$\begin{aligned} \operatorname{area}(M) &= \int_M \omega = \int_{S^1 \times S^1} h^* \omega \\ &= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{2\pi} (a + b\cos(\varphi)) \left( b \sin^2(\varphi) + b\cos^2(\varphi) \right)\, d\theta \wedge d\varphi \\ &= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{2\pi} b(a + b\cos(\varphi))\, d\theta \wedge d\varphi \\ &= 2\pi \int_{\varphi=0}^{2\pi} b(a + b\cos(\varphi))\, d\varphi \\ &= (2\pi)^2 ab. \end{aligned}$$

Chapter 21

# Poincaré Lemma and Thom isomorphism

Manifolds are constructed by gluing local pieces that resemble open subsets of $\mathbb{R}^m$. Therefore, to understand the de Rham cohomology groups of arbitrary manifolds, it suffices to first understand the cohomology groups of these local pieces.

**Theorem 21.1** (Poincaré Lemma). *For $m \geq 0$ we have $H^0(\mathbb{R}^m) = \mathbb{R}$ and all other cohomology groups vanish.*

For instance, the Poincaré Lemma implies that every closed 1-form on $\mathbb{R}^m$ is exact. Starting with

$$\eta = \sum_{i=1}^{m} a_i(x)\, dx_i,$$

and assuming $d\eta = 0$, the following construction defines a smooth function $f\colon \mathbb{R}^m \to \mathbb{R}$ such that $df = \eta$.

It is easy to verify that

$$d\eta = 0 \quad \Leftrightarrow \quad \frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i} \qquad \forall\ 1 \leq i, j \leq m. \tag{21.1}$$

For each $t \in [0,1]$, consider the smooth map

$$\varphi_t\colon \mathbb{R}^m \to \mathbb{R}^m, \quad x \mapsto tx. \tag{21.2}$$

Clearly, $\varphi_1^*\eta = \eta$ and $\varphi_0^*\eta = 0$. Therefore, by the Fundamental Theorem of Calculus,

$$\eta = \varphi_1^*\eta - \varphi_0^*\eta = \int_0^1 \frac{\partial}{\partial t}\varphi_t^*\eta\ dt.$$

By the definition of the pullback, we have

$$\varphi_t^*\eta = \sum_{i=1}^m a_i(tx)\, d(tx_i) = t\sum_{i=1}^m a_i(tx)\, dx_i.$$

Thus,

$$\frac{\partial}{\partial t}\varphi_t^*\eta = \sum_{i=1}^m a_i(tx)\, dx_i + t\sum_{i=1}^m \frac{\partial a_i(tx)}{\partial t}\, dx_i.$$

Applying the chain rule and then using (21.1), the second term becomes

$$\begin{aligned} t\sum_{i=1}^m \frac{\partial a_i(tx)}{\partial t}\, dx_i &= t\sum_{i=1}^m\sum_{j=1}^m \frac{\partial a_i}{\partial x_j}(tx)\, x_j\, dx_i \\ &= t\sum_{i=1}^m\sum_{j=1}^m \frac{\partial a_j}{\partial x_i}(tx)\, x_j\, dx_i \\ &= t\sum_{j=1}^m x_j \sum_{i=1}^m \frac{\partial a_j}{\partial x_i}(tx)\, dx_i \\ &= \sum_{j=1}^m x_j\, da_j(tx). \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t}\varphi_t^*\eta = \sum_{i=1}^m a_i(tx)\, dx_i + \sum_{i=1}^m x_i\, da_i(tx) = d\left(\sum_{i=1}^m x_i\, a_i(tx)\right).$$

Let

$$f_t\colon \mathbb{R}^m \to \mathbb{R}, \qquad f_t(x) = \sum_{i=1}^m x_i\, a_i(tx).$$

Then,

$$\eta = \int_0^1 df_t\ dt.$$

Since $d$ is taken with respect to the $x$-variables and the integral is with respect to the parameter $t$, the two operations commute. Hence,

$$\eta = d\int_0^1 f_t\ dt.$$

Therefore, $\eta = df$, where

$$f(x) = \int_0^1 f_t(x)\ dt.$$

**Remark 21.2.** This proof does not extend to compactly supported 1-forms, because the support of $f_t$, and hence of $f$, is the preimage under $\varphi_t$ of the support of $\eta$, which grows as $t \to 0$.

**Exercise 21.3.** The following is a well-known result that allows us to integrate smooth differential forms over continuous maps into manifolds.

**Theorem 21.4** (Approximation of Continuous Maps by Smooth Maps [**Hir76**, Theorem 2.7])**.** *Let $X$ be a compact $C^\infty$ manifold (with or without boundary), and let $Y$ be a smooth manifold. Then every continuous map $f\colon X \to Y$ is homotopic to a smooth map, and given any open cover $\mathcal{U}$ of $Y$, $f$ can be approximated by a smooth map $g$ such that $f(x)$ and $g(x)$ lie in the same element of $\mathcal{U}$ for all $x \in X$.*

Keeping this in mind, suppose $M$ is a simply-connected smooth manifold and $\eta$ is a closed 1-form on $M$. Fix a base point $p_0 \in M$. For any other point $p \in M$, let $\gamma\colon I \to M$ be any smooth path from $p_0$ to $p$. Here, $I$ is closed interval in $\mathbb{R}$. Show that

$$f\colon M \to \mathbb{R}, \quad f(p) = \int_\gamma \eta := \int_I \gamma^*\eta$$

is well-defined (i.e., it does not depend on the choice of $\gamma$). Prove that $df = \eta$.

There are different ways to prove the Poincaré Lemma. Here, we follow an approach that relies on important and broadly applicable techniques from homological algebra.

Suppose $f\colon M \to M'$ is a smooth map between two manifolds. Since the exterior derivative $d$ commutes with pullback by $f$, the map $f$ induces a linear map

$$f^*\colon H^k(M') \to H^k(M) \qquad \forall\ k \geq 0.$$

However, the same is not true for compactly supported cohomology groups, since the pullback of a compactly supported $k$-form may fail to be compactly supported. This issue is resolved if we assume that $f$ is *proper*, meaning that the preimage of every compact set is compact.

**Definition 21.5.** Two smooth maps $f_0, f_1\colon M \to M'$ are called **smoothly homotopic** if there exists a smooth map

$$F\colon [0,1] \times M \to M'$$

such that

$$F|_{\{0\}\times M} = f_0 \quad \text{and} \quad F|_{\{1\}\times M} = f_1.$$

**Example 21.6.** The identity map $\mathrm{id}\colon \mathbb{R}^m \to \mathbb{R}^m$ and the constant map $0\colon \mathbb{R}^m \to \mathbb{R}^m$ are smoothly homotopic. One such smooth homotopy is given by

$$F\colon [0,1]\times\mathbb{R}^m \to \mathbb{R}^m, \quad (t,x)\mapsto tx,$$

that we used in (21.2).

**Theorem 21.7.** *If $f_0, f_1\colon M \to M'$ are smoothly homotopic, then*

$$f_0^* = f_1^*\colon H^k(M') \to H^k(M).$$

**Corollary 21.8** (Poincaré Lemma)**.** *For $m \geq 0$ we have $H^0(\mathbb{R}^m) = \mathbb{R}$ while all other cohomology groups vanish.*

**Proof.** Apply Theorem 21.9 to $f_1, f_0\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$, where $f_1(x) = x$ and $f_0(x) = 0$. $\square$

To prove Theorem 21.9, we establish a stronger result at the level of differential forms. More generally, consider two cochain complexes:

$$(A^\bullet, d) := \cdots \longrightarrow A_{k-1} \xrightarrow{d} A_k \xrightarrow{d} A_{k+1} \longrightarrow \cdots$$

and

$$(B^\bullet, d) := \cdots \longrightarrow B_{k-1} \xrightarrow{d} B_k \xrightarrow{d} B_{k+1} \longrightarrow \cdots .$$

A **map of cochain complexes** $f\colon (A^\bullet, d) \to (B^\bullet, d)$ is a sequence of maps $f_k\colon A_k \to B_k$ such that the following diagram commutes for all $k$:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{k-1} \arrow[r, "d"] \arrow[d, "f_{k-1}"] & A_k \arrow[r, "d"] \arrow[d, "f_k"] & A_{k+1} \arrow[r, "d"] \arrow[d, "f_{k+1}"] & \cdots \\
\cdots \arrow[r] & B_{k-1} \arrow[r, "d"] & B_k \arrow[r, "d"] & B_{k+1} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

A map of cochain complexes induces maps on cohomology groups. Two such maps

$$f, g\colon (A^\bullet, d) \to (B^\bullet, d)$$

are called **chain homotopic** if there exists a sequence $I = \{I_k\}$ of degree *decreasing* maps $I_k\colon A_k \to B_{k-1}$ such that

$$f_k - g_k = d \circ I_k + I_{k+1} \circ d.$$

This is typically illustrated by the following (non-commutative) diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{k-1} \arrow[r, "d"] \arrow[d, "f_{k-1}-g_{k-1}"'] & A_k \arrow[r, "d"] \arrow[d, "f_k-g_k"] \arrow[dl, "I_k"'] & A_{k+1} \arrow[r, "d"] \arrow[d, "f_{k+1}-g_{k+1}"] \arrow[dl, "I_{k+1}"'] & \cdots \\
\cdots \arrow[r] & B_{k-1} \arrow[r, "d"] & B_k \arrow[r, "d"] & B_{k+1} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

It is a general fact that chain homotopic maps induce the same maps on cohomology; c.f. [**Wei94**, Lemma 1.3.2]. The proof is relatively easy and involves some diagram chasing.

In the setting of smooth maps between manifolds, the following theorem shows that if $f_0, f_1 \colon M \to M'$ are smoothly homotopic, then the induced pullback maps on differential forms are chain homotopic. Consequently, Theorem 21.9 follows.

**Theorem 21.9.**

*(I) For $k \geq 1$, there exists a linear map*

$$h_k \colon \Omega^k([0,1] \times M) \to \Omega^{k-1}(M)$$

*such that*

$$j_1^* - j_0^* = d \circ h_k + h_{k+1} \circ d,$$

*where $j_0, j_1 \colon M \to [0,1] \times M$ are the inclusion maps of the boundary:*

$$j_0(x) = (0, x) \quad \textit{and} \quad j_1(x) = (1, x).$$

*(II) If $f_0, f_1 \colon M \to M'$ are smoothly homotopic, then the pullback maps*

$$f_0^*, f_1^* \colon (\Omega^\bullet(M'), d) \to (\Omega^\bullet(M), d)$$

*are chain homotopic. Therefore, they induce the same map between cohomology groups of $M'$ and $M$.*

**Proof.** For $k \geq 1$, every differential $k$-form on $[0,1] \times M$ can be uniquely decomposed as

$$dt \wedge \alpha + \beta,$$

where $\iota_{\partial_t} \beta = 0$, and in any product chart $[0,1] \times U$, with local coordinates $(x_1, \ldots, x_m)$ on $U$, the forms $\alpha$ and $\beta$ only involve the differentials $dx_i$ (although their coefficients may depend on both $x$ and $t$).

Define

$$h_k(dt \wedge \alpha + \beta) = \int_0^1 \alpha \, dt \in \Omega^{k-1}(M).$$

In other words, the operator $h_k$ integrates the coefficient functions of $\alpha$ with respect to $t$, yielding functions that only depend on $x$.

To verify that this operator has the desired properties, it suffices to work in a local chart on $M$.

In a chart with coordinates $(x_1, \ldots, x_m)$, the form $\alpha$ is a sum of terms of the form

$$a(t, x) \; dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}},$$

and $\beta$ is a sum of terms of the form

$$b(t,x)\ dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

Taking one of these terms at a time for simplicity of notation, we compute:

(1)

$$(j_1^* - j_0^*)(dt \wedge \alpha + \beta) = (b(1,x) - b(0,x))\ \ dx_{j_1} \wedge \cdots \wedge dx_{j_k}$$

(2)

$$\begin{aligned}(d \circ h_k)(dt \wedge \alpha + \beta) &= d_x \left( \left( \int_0^1 a(t,x)\, dt \right)\ dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \right) \\ &= \left( \int_0^1 d_x a(t,x)\, dt \right)\ dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \\ &= \sum_{i_0} \left( \int_0^1 \frac{\partial a(t,x)}{\partial x_{i_0}}\, dt \right)\ dx_{i_0} \wedge \cdots \wedge dx_{i_{k-1}}\end{aligned}$$

(3)

$$\begin{aligned}(h_{k+1} \circ d)(dt \wedge \alpha + \beta) &= h_{k+1} \left( -dt \wedge d_x \alpha + dt \wedge \frac{\partial \beta}{\partial t} + d_x \beta \right) \\ &= h_{k+1} \Big( -dt \wedge \sum_{i_0} \frac{\partial a(t,x)}{\partial x_{i_0}}\ dx_{i_0} \wedge \cdots \wedge dx_{i_{k-1}} \\ &\quad + dt \wedge \frac{\partial b(t,x)}{\partial t}\ dx_{j_1} \wedge \cdots \wedge dx_{j_k} \Big) \\ &= -\sum_{i_0} \left( \int_0^1 \frac{\partial a(t,x)}{\partial x_{i_0}}\, dt \right)\ dx_{i_0} \wedge \cdots \wedge dx_{i_{k-1}} \\ &\quad + \left( \int_0^1 \frac{\partial b(t,x)}{\partial t}\, dt \right)\ dx_{j_1} \wedge \cdots \wedge dx_{j_k} \\ &= -\sum_{i_0} \left( \int_0^1 \frac{\partial a(t,x)}{\partial x_{i_0}}\, dt \right)\ dx_{i_0} \wedge \cdots \wedge dx_{i_{k-1}} \\ &\quad + (b(1,x) - b(0,x))\ \ dx_{j_1} \wedge \cdots \wedge dx_{j_k}\end{aligned}$$

It is clear from the calculations above that

$$(j_1^* - j_0^*)(dt \wedge \alpha + \beta) = (d \circ h_k + h_{k+1} \circ d)\,(dt \wedge \alpha + \beta).$$

This finishes the proof of part I.

For part II, we have $f_0 = F \circ j_0$ and $f_1 = F \circ j_1$. Therefore,

$$f_0^* = j_0^* \circ F^* \quad \text{and} \quad f_1^* = j_1^* \circ F^*.$$

By part I and since pullback commutes with $d$, we have

$$f_1^* - f_0^* = (j_1^* - j_0^*) \circ F^* = (d \circ h_k + h_{k+1} \circ d) \circ F^* = d \circ I_k + I_{k+1} \circ d,$$

where

$$I_k = h_k \circ F^* \colon \Omega^k(M) \to \Omega^{k-1}(M).$$

□

For compactly-supported cohomology, we obtain a result that is formally opposite to Theorem 21.1. This is no coincidence. As we explain later, on any smooth oriented $m$-manifold $M$ without boundary, the bilinear pairing

$$\Omega^k(M, \mathbb{R}) \times \Omega_c^{m-k}(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

defined by integration of the wedge product of forms,

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta,$$

induces a natural isomorphism between $H_c^{m-k}(M, \mathbb{R})$ and the dual of $H^k(M, \mathbb{R})$. In particular, $H^k(M, \mathbb{R})$ and $H_c^{m-k}(M, \mathbb{R})$ are finite-dimensional real vector spaces of the same dimension (This is one version of Poincaré duality).

**Theorem 21.10** (Thom Isomorphism). *For every smooth manifold $M$, we have an isomorphism*

$$H_c^{k+1}(M \times \mathbb{R}, \mathbb{R}) \cong H_c^k(M, \mathbb{R}).$$

*Consequently, for every $m \geq 0$, we have $H_c^m(\mathbb{R}^m, \mathbb{R}) \cong \mathbb{R}$, and all other compactly-supported cohomology groups of $\mathbb{R}^m$ vanish.*

**Proof.** For any smooth manifold $M$, we construct a linear map

$$I \colon \Omega_c^{k+1}(\mathbb{R} \times M) \longrightarrow \Omega_c^k(M) \tag{21.3}$$

with the following properties:

(1) It sends closed forms to closed forms.

(2) It sends exact forms to exact forms.

(3) It is surjective; in fact, there exists a map $J \colon \Omega_c^k(M) \to \Omega_c^{k+1}(\mathbb{R} \times M)$ such that $I \circ J = \mathrm{id}$.

(4) If $\eta \in \Omega_c^{k+1}(\mathbb{R} \times M)$ is closed and $I(\eta)$ is exact, then $\eta$ is exact.

It is easy to see that such an operator $I$ induces an isomorphism

$$H_c^{k+1}(\mathbb{R} \times M, \mathbb{R}) \longrightarrow H_c^k(M, \mathbb{R}).$$

Just as in the proof of the Poincaré Lemma, we decompose

$$\eta = dt \wedge \alpha + \beta,$$

and define

$$I(\eta) = \int_{\mathbb{R}} \alpha \, dt.$$

The support of $I(\eta)$ is the projection to $M$ of the compact support of $\eta$.

Since

$$d\eta = dt \wedge \left(-d_x\alpha + \frac{\partial \beta}{\partial t}\right) + d_x\beta,$$

we conclude that $\eta$ is closed if and only if

$$d_x\alpha = \frac{\partial \beta}{\partial t} \quad \text{and} \quad d_x\beta = 0,$$

where $d_x$ denotes the exterior derivative with respect to the $M$-coordinates in a product chart.

To check (1): If $\eta$ is closed, then

$$d(I(\eta)) = d_x \int_{\mathbb{R}} \alpha \, dt = \int_{\mathbb{R}} d_x\alpha \, dt = \int_{\mathbb{R}} \frac{\partial \beta}{\partial t} \, dt = \beta(\infty, x) - \beta(-\infty, x) = 0.$$

To check (2): If

$$\eta = d\gamma = dt \wedge \left(-d_x a + \frac{\partial b}{\partial t}\right) + d_x b$$

for some $\gamma = dt \wedge a + b \in \Omega_c^k(\mathbb{R} \times M)$, then

$$I(\eta) = \int_{\mathbb{R}} \alpha \, dt = \int_{\mathbb{R}} \left(-d_x a + \frac{\partial b}{\partial t}\right) \, dt = -d_x \int_{\mathbb{R}} a \, dt,$$

which is clearly exact.

To construct a right inverse $J$, choose any compactly supported function $h(t)$ on $\mathbb{R}$ with total integral $\int_{\mathbb{R}} h(t)\, dt = 1$, and define

$$J\colon \Omega_c^k(M) \to \Omega_c^{k+1}(\mathbb{R} \times M), \qquad \alpha \mapsto dt \wedge (h(t)\alpha).$$

It is easy to check that $I \circ J = \mathrm{id}$.

Finally, to verify (4), suppose $\eta = dt \wedge \alpha + \beta \in \Omega_c^{k+1}(\mathbb{R} \times M)$ is closed and $I(\eta)$ is exact. Then $J(I(\eta))$ is also exact, and $I(\eta - J(I(\eta))) = 0$. So by replacing $\eta$ with $\eta - J(I(\eta))$, we may assume $I(\eta) = 0$. In this case, define

$$b(t, x) = \int_{-\infty}^{t} \alpha(s, x)\, ds.$$

It is easy to check that $b \in \Omega_c^k(\mathbb{R} \times M)$ satisfies $d\gamma = \eta$.

This completes the proof of the first statement in Theorem 21.10. The second statement then follows by induction on $m$, using the first statement. The base case $m = 1$ was addressed in Exercise 18.9. $\square$

Both the Poincaré Lemma and Thom Isomorphism hold in more general settings. Suppose $\pi\colon E \to M$ is a smooth vector bundle of rank $r$. Then,

$$H^k(E, \mathbb{R}) \cong H^k(M, \mathbb{R}). \tag{21.4}$$

Furthermore, if $E$ is oriented, integration along the fibers of $E$ defines an isomorphism

$$H_c^k(E, \mathbb{R}) \longrightarrow H_c^{k-r}(M, \mathbb{R}). \tag{21.5}$$

These generalizations are particularly useful for relating singular homology (which we do not study in this book) and de Rham cohomology.

**Exercise 21.11.** Use Theorem 21.9 to prove (21.4). Then, by considering local trivializations of $E$ and verifying that the isomorphism constructed in the proof of Theorem 21.10 is compatible with the transition maps, prove (21.5).

**Exercise 21.12.** For $m \geq 1$, suppose $M$ is a compact, connected, orientable $m$-dimensional submanifold of $\mathbb{R}^{m+1}$. In Exercise 15.4, we showed that the normal bundle of $M$ is trivial. Use this, together with the fact (which we do not prove in this book) that a neighborhood of any submanifold is diffeomorphic to a neighborhood of the zero section in its normal bundle, to prove that $\mathbb{R}^{m+1} \setminus M$ has exactly two connected components.

## Solutions to exercises

**Exercise 21.3.** First, we show that $\int_\gamma \eta$ does not depend on the choice of path $\gamma$. Suppose $\gamma_1$ and $\gamma_2$ are two smooth paths from the fixed base point $p_0$ to a point $p \in M$. Then the concatenation of $\gamma_1$ with the reverse of $\gamma_2$ (e.g., $\gamma_2(1-t)$ when $t \in I = [0,1]$) defines a continuous loop

$$\gamma\colon S^1 \to M.$$

Since $M$ is simply connected, the map $\gamma$ extends to a continuous map

$$\widetilde{\gamma}\colon D^2 \to M,$$

where $D^2$ is the 2-dimensional disk bounding $S^1$. By Theorem 21.4, for any $\varepsilon > 0$, the map $\widetilde{\gamma}$ can be approximated by smooth maps $\widetilde{\gamma}_\varepsilon\colon D^2 \to M$ that are $\varepsilon$-close to $\widetilde{\gamma}$ in the uniform $C^0$-norm. Since $\eta$ is closed, and by Stokes' Theorem,

$$0 = \int_{D^2} \widetilde{\gamma}_\varepsilon^* d\eta = \int_{S^1} \gamma_\varepsilon^* \eta,$$

where $\gamma_\varepsilon$ is the restriction of $\widetilde{\gamma}_\varepsilon$ to $S^1$. Letting $\varepsilon \to 0$ and using continuity of integration, we conclude that

$$0 = \int_{S^1} \gamma^* \eta = \int_{\gamma_1} \eta - \int_{\gamma_2} \eta.$$

To show that $df = \eta$ at a point $p \in M$, note that the derivative is a local property, so we may work in a coordinate chart around $p$. Let $(x_1, \dots, x_m)$ be local coordinates on a neighborhood $U$ of $p$, identifying $p$ with the origin $0 \in U \subset \mathbb{R}^m$. Then

$$\frac{\partial f}{\partial x_i}(0) = \lim_{h \to 0} \frac{f(he_i) - f(0)}{h},$$

where $e_i$ is the standard unit vector in the $i$-th direction. Fix a path $\gamma_0$ from $p_0$ to $p$. Concatenating it with the straight-line path

$$\gamma_h\colon [0,1] \to U, \qquad t \mapsto the_i,$$

gives a path from $p_0$ to the point corresponding to $he_i$. Therefore,

$$\frac{f(he_i) - f(0)}{h} = \frac{1}{h} \int_0^1 \gamma_h^* \eta.$$

Suppose

$$\eta = \sum_{j=1}^m a_j(x)\, dx_j.$$

Then

$$\gamma_h^* \eta = a_i(t\, h\, e_i)\, h\, dt,$$

and so

$$\frac{1}{h}\gamma_h^*\eta = a_i(t\,h\,e_i)\,dt.$$

Therefore,

$$\frac{f(he_i) - f(0)}{h} = \int_0^1 a_i(t\,h\,e_i)\,dt,$$

and taking the limit as $h \to 0$ gives

$$\lim_{h\to 0}\frac{f(he_i) - f(0)}{h} = \int_0^1 a_i(0)\,dt = a_i(0).$$

We conclude that

$$df|_0 = \sum_i \frac{\partial f}{\partial x_i}(0)\,dx_i = \eta|_0.$$

□

**Exercise 21.11.** The map

$$F\colon [0,1]\times E \longrightarrow E, \quad (t,v) \longrightarrow tv$$

is a smooth homotopy interpolating between the identity map on $E$ and the projection map $\pi\colon E \longrightarrow M$. Note that $\pi$ is viewed as a map from $E$ to itself, with image equal to $M$. From this perspective, the map

$$\pi^*\colon H^k(E,\mathbb{R}) \longrightarrow H^k(E,\mathbb{R})$$

acts by first restricting a cohomology class to $M$ and then pulling it back to $E$. By Theorem 21.9, $\pi^*$ is the identity map on $H^k(E,\mathbb{R})$. Therefore, every cohomology class on $E$ is equal to the pullback of its restriction to $M$, and we conclude that

$$H^k(E,\mathbb{R}) \cong H^k(M,\mathbb{R}).$$

Consider a collection of local trivializations

$$\{\Phi_i\colon E|_{U_i} \longrightarrow U_i \times \mathbb{R}^r\}$$

over a countable open cover $\{U_i\}$ of $M$, compatible with the orientation on $E$.

For each $i$, let

$$I_i\colon \Omega_c^{k+r}(U_i\times\mathbb{R}^r) \longrightarrow \Omega_c^k(U_i)$$

be the map obtained by repeatedly applying (21.3) $r$ times, eliminating one copy of $\mathbb{R}$ at a time. More precisely, any $\eta \in \Omega_c^{k+r}(U_i\times\mathbb{R}^r)$ can be written as

$$\eta = dt_1\wedge\cdots\wedge dt_r\wedge\alpha + \beta,$$

where $(t_1,\ldots,t_r)$ are coordinates on $\mathbb{R}^r$, and $\beta$ consists of terms that do not include $dt_1\wedge\cdots\wedge dt_r$. Then,

$$I_i(\eta) = \int_{\mathbb{R}^r}\alpha\;dt_1\cdots dt_r.$$

The support of $I_i(\eta)$ is the projection to $U_i$ of the compact support of $\eta$. Using a partition of unity subordinate to the open cover $\{U_i\}$, every $\eta \in \Omega^{k+r}(E)$ can be decomposed as a countable sum $\sum_i \eta_i$, where each $\eta_i$ is supported in $E|_{U_i}$. Under the identifications of $\Omega_c^{k+r}(U_i \times \mathbb{R}^r)$ and $\Omega_c^{k+r}(E|_{U_i})$ given by $\Phi_i$, we define

$$I(\eta) = \sum_i I_i(\eta_i).$$

To prove that $I$ is well-defined – i.e., independent of the choice of local trivializations and partition of unity – we must show that different local trivializations

$$\Phi_1 \colon E|_U \longrightarrow U \times \mathbb{R}^r \quad \text{and} \quad \Phi_2 \colon E|_U \longrightarrow U \times \mathbb{R}^r$$

result in the same maps

$$I_1, I_2 \colon \Omega_c^{k+r}(U \times \mathbb{R}^r) \longrightarrow \Omega_c^k(U).$$

In other words, the change-of-trivialization map (which is a diffeomorphism)

$$\begin{aligned} &\Psi \colon U \times \mathbb{R}^r \longrightarrow U \times \mathbb{R}^r, \\ &(x, t = (t_1, \dots, t_r)) \mapsto (x, s = (s_1, \dots, s_r)) = (x, \Phi_{1\mapsto 2}(x)t) \end{aligned}$$

satisfies

$$I_2(\eta) = I_1(\Psi^*\eta) \qquad \forall\, \eta \in \Omega_c^{k+r}(U \times \mathbb{R}^r).$$

If

$$\eta = ds_1 \wedge \cdots \wedge ds_r \wedge \alpha(x,s) + \beta(x,s),$$

then

$$\Psi^*\eta = \det(\Phi_{1\mapsto 2}(x))\, dt_1 \wedge \cdots \wedge dt_r \wedge \alpha(x, \Phi_{1\mapsto 2}(x)t) + \beta'(x,t),$$

where $\beta'(x,t)$ collects all terms that do not include $dt_1 \wedge \cdots \wedge dt_r$, and $\alpha(x, \Phi_{1\mapsto 2}(x)t)$ denotes the $k$-form whose coefficients are the coefficients of $\alpha$ written as functions of $(x,t)$ under the change of variables.

Therefore,

$$I_2(\eta) = \int_{\mathbb{R}^r} \alpha_2(x,s)\; ds_1 \cdots ds_r$$

and

$$I_1(\Psi^*\eta) = \int_{\mathbb{R}^r} \det(\Phi_{1\mapsto 2}(x))\, \alpha_2(x, \Phi_{1\mapsto 2}(x)t)\; dt_1 \cdots dt_r.$$

Since the local trivializations are compatible with the orientation we have $\det(\Phi_{1\mapsto 2}(x)) > 0$. For each fixed $x$, the map $t \mapsto \Phi_{1\mapsto 2}(x)t$ is a linear transformation with Jacobian determinant $\det(\Phi_{1\mapsto 2}(x))$. It follows from the **change of variables formula** in multivariable calculus that the two integrals are equal. $\square$

**Exercise 21.12.** As explained in the question, a neighborhood $U$ of $M$ in $\mathbb{R}^{m+1}$ can be identified (via a diffeomorphism) with $V = M \times (-1, 1)$, identifying $M$ with $M \times \{0\}$. Clearly, $V \setminus M$ has two connected components, distinguished by the sign of the variable in $\mathbb{R}^*$.

First, we show that $\mathbb{R}^{m+1} \setminus M$ has at least two connected components. Fix $p \in M$, choose $0 < \varepsilon < 1$, and let $p_\pm = (p, \pm\varepsilon) \in V$. We denote the corresponding points in $U$ by the same letters. Suppose $p_\pm$ belong to the same connected component of $\mathbb{R}^{m+1} \setminus M$. Fix a smooth path $\gamma$ connecting them. Attaching to $\gamma$ the straight path from $p_-$ to $p_+$ in $V$, which intersects $M$ transversely at $(p, 0)$, we obtain a loop

$$\widetilde{\gamma}\colon S^1 \longrightarrow \mathbb{R}^{m+1}$$

that intersects $M$ transversely once at $p \in M$.

Suppose $h(t)$ is a smooth function compactly supported in $(-\varepsilon, \varepsilon)$ with non-zero integral

$$\int_{-\varepsilon}^{\varepsilon} h(t)\, dt = c \neq 0.$$

Under the identification of $U$ and $V$, the compactly supported closed 1-form $\eta = h(t)\, dt$ on $V$ defines a compactly supported closed 1-form $\eta$ on $\mathbb{R}^{m+1}$. It is clear that

$$\int_{\widetilde{\gamma}} \eta = \pm c.$$

On the other hand, by the Thom isomorphism or Poincaré lemma, $\eta$ is exact; i.e., $\eta = df$ for some smooth function $f$. Therefore, by Stokes' theorem,

$$\int_{\widetilde{\gamma}} \eta = \int_{S^1} d(f \circ \widetilde{\gamma}) = 0.$$

This is a contradiction. We conclude that the images $U_\pm$ of $M \times (0, 1)$ and $M \times (-1, 0)$ in $U$ belong to two different connected components of $\mathbb{R}^{m+1} \setminus M$.

Since $\mathbb{R}^{m+1}$ is connected, every point in $\mathbb{R}^{m+1} \setminus M$ can be connected to a point in $U_\pm$ without intersecting $M$. We conclude that $\mathbb{R}^{m+1} \setminus M$ has exactly two connected components. Since $M$ is compact, one of these components is unbounded, and the other must be bounded by $M$. $\square$

Chapter 22

# Mayer-Vietoris sequence

As we have emphasized repeatedly, manifolds are constructed by gluing together local pieces that resemble open subsets of $\mathbb{R}^m$. The Poincaré Lemma and the Thom isomorphism illustrate that the cohomology of these local pieces is relatively simple. To deduce information about the de Rham cohomology of a general manifold from that of its local pieces, we need an inductive tool. The Mayer-Vietoris sequence provides precisely such a mechanism, arising naturally from basic principles of homological algebra.

Suppose $M = U_1 \cup U_2$ is a decomposition of the smooth $m$-manifold $M$ into two open subsets. For every $k \geq 0$ and $i = 1, 2$, let

$$R_i \colon \Omega^k(M) \longrightarrow \Omega^k(U_i) \quad \text{and} \quad r_i \colon \Omega^k(U_i) \longrightarrow \Omega^k(U_1 \cap U_2)$$

denote the restriction maps. Putting them together appropriately gives us a sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{R=R_1 \oplus R_2} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{r=r_1-r_2} \Omega^k(U_1 \cap U_2) \longrightarrow 0.$$

It is easy to verify that this is a short exact sequence; that is, $R$ is injective, $r$ is surjective, and $\ker(r) = \mathrm{Im}(R)$. Surjectivity of $r$ can be proved using a partition of unity subordinate to the covering.

Running over all $k \geq 0$, the maps $R$ and $r$ define a short exact sequence of cochain complexes, i.e., a short exact sequence (in vertical direction)

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{k-1}(M) \arrow[r, "d"] \arrow[d, "R"] & \Omega^{k}(M) \arrow[r, "d"] \arrow[d, "R"] & \Omega^{k+1}(M) \arrow[r, "d"] \arrow[d, "R"] & \cdots \\
\cdots \arrow[r] & \Omega^{k-1}(U_1 \sqcup U_2) \arrow[r, "d"] \arrow[d, "r"] & \Omega^{k}(U_1 \sqcup U_2) \arrow[r, "d"] \arrow[d, "r"] & \Omega^{k+1}(U_1 \sqcup U_2) \arrow[r, "d"] \arrow[d, "r"] & \cdots \\
\cdots \arrow[r] & \Omega^{k-1}(U_1 \cap U_2) \arrow[r, "d"] & \Omega^{k}(U_1 \cap U_2) \arrow[r, "d"] & \Omega^{k+1}(U_1 \cap U_2) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

between the de Rham cochain complexes of $M$, $U_1 \sqcup U_2$, and $U_1 \cap U_2$. *It is a basic theorem in homological algebra that a short exact sequence of cochain complexes results in a long exact sequence of cohomology groups*; see [**Wei94**, Theorem 1.3.1]. In our case, this short exact sequence yields the long exact sequence of cohomology groups:

$$\begin{tikzcd}
0 \arrow[r] & H^0(M,\mathbb{R}) \arrow[r, "R"] & H^0(U_1,\mathbb{R}) \oplus H^0(U_2,\mathbb{R}) \arrow[r, "r"] & H^0(U_1 \cap U_2,\mathbb{R}) \arrow[dll, "\delta"] \\
& H^1(M,\mathbb{R}) \arrow[r] & H^1(U_1,\mathbb{R}) \oplus H^1(U_2,\mathbb{R}) \arrow[r] & H^1(U_1 \cap U_2,\mathbb{R}) \arrow[dll, "\delta"] \\
& \cdots & & \cdots \arrow[dll, "\delta"] \\
& H^m(M,\mathbb{R}) \arrow[r] & H^m(U_1,\mathbb{R}) \oplus H^m(U_2,\mathbb{R}) \arrow[r] & H^m(U_1 \cap U_2,\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

The maps in each row are induced by the maps $R$ and $r$. We now describe explicitly the so-called **connecting homomorphisms**

$$\delta\colon H^k(U_1 \cap U_2,\mathbb{R}) \longrightarrow H^{k+1}(M,\mathbb{R})$$

in our setting. This long exact sequence is known as the **Mayer-Vietoris** sequence.

Any cohomology class $[\eta] \in H^k(U_1 \cap U_2,\mathbb{R})$ is represented by a closed $k$-form

$$\eta \in \Omega^k(U_1 \cap U_2,\mathbb{R}).$$

Suppose $\{\varrho_i\colon U_i \to [0,1]\}$ is a partition of unity subordinate to the open cover $\{U_1, U_2\}$. Then the $k$-forms

$$\eta_1 = \varrho_2\eta \quad \text{and} \quad \eta_2 = -\varrho_1\eta$$

on $U_1 \cap U_2$ can be trivially (i.e., by zero) extended to well-defined $k$-forms on $U_1$ and $U_2$ such that $r(\eta_1,\eta_2) = \eta$. Let $\widetilde{\eta}_i = d\eta_i$. Since $d\varrho_1 + d\varrho_2 = 0$, we have

$$\widetilde{\eta}_1|_{U_1 \cap U_2} = \widetilde{\eta}_2|_{U_1 \cap U_2}.$$

Therefore, $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ define a $(k+1)$-form $\widetilde{\eta}$ on $M$. Moreover, this form is closed since each $\widetilde{\eta}_i$ is exact. Hence, it defines a cohomology class $[\widetilde{\eta}]$, which we define to be $\delta([\eta])$. It takes a lengthy but straightforward calculation to check that this result is independent of the choices involved. More precisely, to show that $\delta$ is well-defined, one must verify the following:

(1) If $\eta$ is replaced by a cohomologous form (i.e., changed by an exact form), then $\widetilde{\eta}$ also changes by an exact form.

(2) If the partition of unity is changed, the resulting form $\widetilde{\eta}$ again changes by an exact form.

The reader is encouraged to verify these properties as an instructive exercise.

**Example 22.1.** The domain of the 2-chart atlas (2.3) gives a decomposition $S^2 = U_1 \cup U_2$, where $U_i \cong \mathbb{R}^2$ and $U_1 \cap U_2 = \mathbb{R}^2 \setminus \{0\} \cong S^1 \times \mathbb{R}$. By the Poincaré Lemma (see Exercise 21.11),

$$H^k(U_1 \cap U_2, \mathbb{R}) \cong H^k(S^1, \mathbb{R}).$$

Therefore, the Mayer–Vietoris sequence takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r, "R"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "r"] & \mathbb{R} \arrow[dll, "\delta"] \\
& H^1(S^2, \mathbb{R}) \arrow[r] & 0 \oplus 0 \arrow[r] & \mathbb{R} \arrow[dll, "\delta"] \\
& H^2(S^2, \mathbb{R}) \arrow[r] & 0 \oplus 0 \arrow[r] & 0
\end{tikzcd}$$

It is straightforward to verify that the first row is exact; that is, the map

$$H^0(U_1, \mathbb{R}) \oplus H^0(U_2, \mathbb{R}) \to H^0(U_1 \cap U_2, \mathbb{R})$$

is surjective. Therefore, $H^1(S^2, \mathbb{R}) = 0$ and $H^2(S^2, \mathbb{R}) \cong H^1(S^1, \mathbb{R}) \cong \mathbb{R}$ is one-dimensional.

The second cohomology class in $H^2(S^2, \mathbb{R})$ corresponding to $[d\theta] \in H^1(S^1, \mathbb{R})$ under the connecting homomorphism (which is an isomorphism in this case) can be explicitly described as follows.

Starting from $\eta = d\theta \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$, we construct $\eta_1$ and $\eta_2$ on $U_1$ and $U_2$ by multiplying $d\theta$ with suitable functions. Writing down explicit bump functions with compact support is difficult because such functions are not analytic. However, all we really need are functions $f_1, f_2$ such that $\eta_1 = f_2\eta$, $\eta_2 = -f_1\eta$, and $f_1 + f_2 = 1$, with $f_i$ vanishing at the origin of $U_i$.

The chart maps in (2.9) equip $U_i$ with polar coordinates $(r_i, \theta_i)$ such that $r_1 r_2 = 1$ and $\theta_1 + \theta_2 = 0$. In complex coordinates, this corresponds to $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, related by $z_2 = \frac{1}{z_1}$.
Let us choose the angular coordinate $\theta$ on $U_1 \cap U_2$ to be $\theta_1$. The functions

$$f_2(r_1) = \frac{r_1^2}{1+r_1^2} \quad \text{and} \quad f_1(r_2) = \frac{r_2^2}{1+r_2^2}$$

satisfy the required properties. We obtain

$$\eta_1 = \frac{r_1^2}{1+r_1^2}\, d\theta_1 \quad \text{and} \quad \eta_2 = \frac{r_2^2}{1+r_2^2}\, d\theta_2.$$

The resulting 2-forms

$$\widetilde{\eta}_1 = d\eta_1 = \frac{2r_1}{(1+r_1^2)^2}\, dr_1 \wedge d\theta_1 \quad \text{and} \quad \widetilde{\eta}_2 = d\eta_2 = \frac{2r_2}{(1+r_2^2)^2}\, dr_2 \wedge d\theta_2$$

are compatible on the overlap and glue together to define a global 2-form $\widetilde{\eta}$ on $S^2$. This form represents a generator $[\widetilde{\eta}]$ of $H^2(S^2, \mathbb{R})$.
One can check that $\widetilde{\eta}$ is, in fact, the standard area form on $S^2$.

The restriction of a compactly supported differential form to an open subset is not necessarily compactly supported. The inclusion map, however, is well-defined. Therefore, corresponding to an open decomposition $M = U_1 \cup U_2$ we get short exact sequences of compactly supported differential forms

$$0 \longrightarrow \Omega_c^k(U_1 \cap U_2) \xrightarrow{\iota = \iota_1 \oplus \iota_2} \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) \xrightarrow{j = j_1 - j_2} \Omega_c^k(M) \longrightarrow 0.$$

in reverse direction. Here, for $i = 1, 2$,

$$\iota_i \colon \Omega_c^k(U_1 \cap U_2) \longrightarrow \Omega_c^k(U_i) \quad \text{and} \quad j_i \colon \Omega_c^k(U_i) \longrightarrow \Omega_c^k(M)$$

are the inclusion maps.

Running over all $k \geq 0$, the maps $\iota$ and $j$ define a short exact sequence of cochain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_c^{k-1}(U_1 \cap U_2) \arrow[r, "d"] \arrow[d, "\iota"] & \Omega_c^k(U_1 \cap U_2) \arrow[r, "d"] \arrow[d, "\iota"] & \Omega_c^{k+1}(U_1 \cap U_2) \arrow[r, "d"] \arrow[d, "\iota"] & \cdots \\
\cdots \arrow[r] & \Omega_c^{k-1}(U_1 \sqcup U_2) \arrow[r, "d"] \arrow[d, "j"] & \Omega_c^k(U_1 \sqcup U_2) \arrow[r, "d"] \arrow[d, "j"] & \Omega_c^{k+1}(U_1 \sqcup U_2) \arrow[r, "d"] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & \Omega_c^{k-1}(M) \arrow[r, "d"] & \Omega_c^k(M) \arrow[r, "d"] & \Omega_c^{k+1}(M) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Consequently, we obtain the long exact sequence of compactly supported cohomology groups:

$$
\begin{aligned}
0 &\longrightarrow H^0(U_1 \cap U_2, \mathbb{R}) \xrightarrow{\iota} H^0_c(U_1, \mathbb{R}) \oplus H^0_c(U_2, \mathbb{R}) \xrightarrow{j} H^0_c(M, \mathbb{R}) \\
&\xrightarrow{\delta} H^1_c(U_1 \cap U_2, \mathbb{R}) \longrightarrow H^1_c(U_1, \mathbb{R}) \oplus H^1_c(U_2, \mathbb{R}) \longrightarrow H^1_c(M, \mathbb{R}) \\
&\xrightarrow{\delta} \cdots \\
&\xrightarrow{\delta} H^m_c(U_1 \cap U_2, \mathbb{R}) \longrightarrow H^m_c(U_1, \mathbb{R}) \oplus H^m_c(U_2, \mathbb{R}) \longrightarrow H^m_c(M, \mathbb{R}) \to 0
\end{aligned}
$$

**Exercise 22.2.** Since $S^2$ is closed, $H^k(S^2, \mathbb{R}) = H^k_c(S^2, \mathbb{R})$. Altering Example 22.1, use the compactly supported version of the Mayer-Vietoris sequence to calculate the cohomology groups of $S^2$.

## Solutions to exercises

**Exercise 22.2.** By the Thom isomorphism,

$$H^k_c(U_i) = \begin{cases} 0 & \text{if } k = 0, 1, \\ \mathbb{R} & \text{if } k = 2 \end{cases}$$

and

$$H^k_c(U_1 \cap U_2) \cong H^{k-1}_c(S^1) = \begin{cases} 0 & \text{if } k = 0, \\ \mathbb{R} & \text{if } k = 1, 2. \end{cases}$$

Therefore, the compactly supported Mayer–Vietoris sequence for $S^2 = \mathbb{R}^2 \cup \mathbb{R}^2$ reads

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, "\iota"] & 0 \oplus 0 \arrow[r, "j"] & H^0(S^2, \mathbb{R}) \arrow[dll, "\delta"] \\
& \mathbb{R} \arrow[r] & 0 \oplus 0 \arrow[r] & H^1(S^2, \mathbb{R}) \arrow[dll, "\delta"] \\
& \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r] & H^2(S^2, \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

Thus, $H^0(S^2, \mathbb{R}) \cong \mathbb{R}$, and the remaining part of the sequence is

$$0 \longrightarrow H^1(S^2, \mathbb{R}) \xrightarrow{\delta} H^2_c(U_1 \cap U_2, \mathbb{R}) \longrightarrow H^2_c(U_1, \mathbb{R}) \oplus H^2_c(U_2, \mathbb{R}) \longrightarrow H^2(S^2, \mathbb{R}) \longrightarrow 0.$$

The map

$$H^2_c(U_1 \cap U_2, \mathbb{R}) \longrightarrow H^2_c(U_1, \mathbb{R})$$

is injective; therefore, $H^1(S^2, \mathbb{R}) = 0$. To see this, it can be observed from the proof of the Thom isomorphism that a generator of $H^2_c(U_1 \cap U_2, \mathbb{R}) = \mathbb{R}$ is of the form

$$h(r)\, dr \wedge d\theta,$$

where $(r, \theta)$ are polar coordinates on $U_1 \cap U_2 = \mathbb{R}^2 \setminus \{0\}$, and $h(r)$ is a function supported near the unit circle $r = 1$, satisfying $\int_0^\infty h(r)\, dr \neq 0$. By Stokes' theorem, since

$$\int_{\mathbb{R}^2} h(r)\, dr \wedge d\theta \neq 0,$$

this form defines a nontrivial class in $H^2_c(\mathbb{R}^2, \mathbb{R})$ as well.

Having shown that $H^1(S^2, \mathbb{R}) = 0$, we conclude that the remainder of the sequence is a short exact sequence

$$0 \longrightarrow H^2_c(U_1 \cap U_2, \mathbb{R}) \longrightarrow H^2_c(U_1, \mathbb{R}) \oplus H^2_c(U_2, \mathbb{R}) \longrightarrow H^2(S^2, \mathbb{R}) \longrightarrow 0,$$

which shows that $H^2(S^2, \mathbb{R}) \cong \mathbb{R}$. We will study top-degree cohomology in more detail in the next section. □

Chapter 23

# Cohomology in top degree

Cohomology in degree zero of any connected manifold $M$ is simply $H^0(M,\mathbb{R}) = \mathbb{R}$. In this lecture, we will see that the top-degree cohomology of manifolds is also simple and often identifiable with $\mathbb{R}$ via integration.

**Theorem 23.1.** *Suppose $M$ is a smooth oriented connected $m$-manifold without boundary. Then $H_c^m(M,\mathbb{R}) \cong \mathbb{R}$. Moreover, integration of compactly supported $m$-forms descends to the desired isomorphism, and for any open subset $U \subset M$, every class in $H_c^m(M,\mathbb{R})$ has a representative supported in $U$.*

**Proof.** Suppose $M$ is oriented. By Stokes' Theorem, the integration map

$$\int_M : \Omega_c^m(M) \longrightarrow \mathbb{R}$$

vanishes on exact forms (note that every $m$-form is automatically closed). Hence, $\int_M$ descends to a well-defined linear map

$$\int_M : H_c^m(M,\mathbb{R}) \longrightarrow \mathbb{R}.$$

This map is surjective because any positive function times a volume form yields a non-zero integral. We aim to show that it is an isomorphism.

Let $\{U_\alpha\}$ be a countable open cover of $M$ by sets homeomorphic to open balls in $\mathbb{R}^m$. Given $\omega \in \Omega_c^m(M)$, a partition of unity lets us write $\omega = \sum_\alpha \omega_\alpha$,

where each $\omega_\alpha$ is compactly supported in $U_\alpha$. Define
$$c_\alpha := \int_M \omega_\alpha.$$
Now fix any open subset $U \subset M$ homeomorphic to an open ball in $\mathbb{R}^m$, and choose $\widetilde{\omega} \in \Omega_c^m(U)$ such that $\int_M \widetilde{\omega} = 1$. This can be done by taking an appropriate multiple of a volume form. We claim that
$$[\omega_\alpha] = c_\alpha[\widetilde{\omega}] \in H_c^m(M, \mathbb{R}).$$
It follows that
$$[\omega] = \Big(\sum_\alpha c_\alpha\Big)[\widetilde{\omega}],$$
i.e., $H_c^m(M, \mathbb{R})$ is generated by a single element.

To verify this claim, fix a finite chain of overlapping open sets $W_0, W_1, \ldots, W_n$, each homeomorphic to an open ball in $\mathbb{R}^m$, such that $W_0 = U$ and $W_n = U_\alpha$. For each $1 \le i \le n$, choose an open set $W_i' \subset W_i \cap W_{i-1}$, also homeomorphic to an open ball in $\mathbb{R}^m$, and pick $\widetilde{\omega}_i \in \Omega_c^m(W_i')$ such that $\int_M \widetilde{\omega}_i = 1$. By the Thom isomorphism,
$$H_c^m(W_i) \cong H_c^m(W_i') \cong \mathbb{R},$$
generated by any $m$-form with non-zero integral. Moving inductively from $W_{i-1}$ to $W_i'$, and from $W_i'$ to $W_i$, we conclude:

- Initially, $\widetilde{\omega} - \widetilde{\omega}_1 = df_0$ for some compactly supported function $f_0$ on $W_0 = U$;
- Then, for each $i = 1, \ldots, n-1$, we have $\widetilde{\omega}_{i+1} - \widetilde{\omega}_i = df_i$ for some compactly supported function $f_i$ on $W_i$;
- Finally, $c\widetilde{\omega}_n - \omega_\alpha = df_n$ for some compactly supported function $f_n$ on $W_n = U_\alpha$.

Adding these relations yields
$$c\widetilde{\omega} - \omega_\alpha = df,$$
for some compactly supported function $f = f_n + c\sum_{i=0}^{n-1} f_i$ on $M$. □

**Remark 23.2.** For any connected smooth manifold $M$, there exists a two-sheeted covering space $\widetilde{M} \to M$ whose total space $\widetilde{M}$ is oriented, and such that the deck transformation acts by reversing orientation. This oriented double cover exists whether or not $M$ is orientable, and it is connected if and only if $M$ is non-orientable.

If $M$ is a smooth connected $m$-manifold without boundary which is not orientable, the connected orientable two-sheeted cover $\widetilde{M}$ can be used to show that $H_c^m(M,\mathbb{R}) = 0$, because the $\mathbb{Z}_2$-action on $H_c^m(\widetilde{M},\mathbb{R})$ is by multiplication by $-1$. Therefore, there are no $\mathbb{Z}_2$-invariant elements in $\Omega_c^m(\widetilde{M})$ with nonzero integral.

**Exercise 23.3.** Suppose $f\colon M \to M'$ is a smooth map between two $m$-dimensional closed (i.e., compact and without boundary) oriented manifolds and $M'$ is connected. The **degree of** $f$ is the number

$$\deg(f) = \frac{\int_M f^*\omega}{\int_{M'} \omega}$$

where $\omega$ is any $m$-form on $M'$ such that $0 \neq [\omega] \in H^m(M')$ such as the volume form of $M'$.

- Prove that $\deg(f)$ is independent of the choice of $\omega$.
- Sard's Theorem [**Hir76**, Ch. 3] shows that a generic point in $M'$ is a regular value of $f$. Use this to show that $\deg(f)$ is an integer that counts the number of preimages of a regular value, with signs.

**Exercise 23.4.** Show that every degree $d$ polynomial $p(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_0$ defines a holomorphic map $p\colon \mathbb{CP}^1 \cong S^2 \to \mathbb{CP}^1$ of degree $d$.

**Exercise 23.5.** Consider two disjoint embeddings of the circle $S^1$ into $\mathbb{R}^3$:

$$\gamma_1, \gamma_2\colon S^1 \hookrightarrow \mathbb{R}^3,$$

with image curves denoted by $C_1$ and $C_2$. Define

$$f\colon S^1 \times S^1 \to S^2, \qquad (p_1, p_2) \mapsto \frac{\gamma_1(p_1) - \gamma_2(p_2)}{|\gamma_1(p_1) - \gamma_2(p_2)|}.$$

The **linking number** $\ell(C_1, C_2)$ of $C_1$ and $C_2$ is defined to be the degree of the map $f$, where each $S^1$ is oriented counterclockwise, and $S^1 \times S^1$ is given the product orientation. Prove that:

- $\ell(C_1, C_2)$ is symmetric in its inputs.
- If $C_1$ bounds an embedded oriented surface $\Sigma$ transverse to $C_2$, then $\ell(C_1, C_2)$ equals the signed count of the intersection points of $D$ and $C_2$.

Theorem 23.1 can be used to prove an interesting duality between ordinary and compactly supported cohomology groups, extending the duality observed by comparing the Poincaré Lemma and the Thom isomorphism on $\mathbb{R}^m$.

**Lemma 23.6.** *On any smooth oriented $m$-manifold $M$ without boundary, the bilinear map*

$$\Omega^k(M,\mathbb{R}) \times \Omega_c^{m-k}(M,\mathbb{R}) \longrightarrow \mathbb{R}$$

*defined by integration of the wedge product of forms,*

$$(\alpha,\beta)\mapsto\langle\alpha,\beta\rangle:=\int_M \alpha\wedge\beta,$$

*descends to a bilinear map*

$$\langle -,-\rangle : H^k(M)\times H_c^{m-k}(M)\longrightarrow\mathbb{R}. \tag{23.1}$$

**Proof.** This is a consequence of Stokes's Theorem and the assumption that $\partial M=\emptyset$. □

**Definition 23.7.** An open cover $\{U_\alpha\}$ of an $m$-manifold $M$ (without boundary) is called a **good cover** if every nonempty finite intersection $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$ is diffeomorphic to $\mathbb{R}^m$. A manifold that admits a finite good cover is said to be of **finite type**.

Every smooth manifold admits a good cover, for example by sufficiently small balls with respect to a Riemannian metric. Every closed manifold is of finite type. Being of finite type implies, for instance, that the (co)homology groups of $M$ are finite-dimensional. One typically restricts to manifolds of finite type to avoid pathological examples such as the infinite-genus surface in Figure 1.

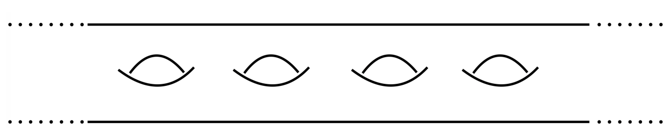

**Figure 1.** An open surface of infinite genus

**Theorem 23.8.** *Suppose $M$ is an oriented smooth $m$-manifold of finite type (without boundary). Then, the pairing* (23.1) *is non-degenerate; that is,*

- $\langle[\alpha],-\rangle=0\Rightarrow[\alpha]=0\in H^k(M,\mathbb{R})$;
- $\langle -,[\beta]\rangle=0\Rightarrow[\beta]=0\in H_c^{m-k}(M,\mathbb{R})$.

*Consequently,*

$$H^k(M,\mathbb{R})\cong H_c^{m-k}(M,\mathbb{R})^* \qquad \forall\ k. \tag{23.2}$$

Since every real vector space is (non-canonically) isomorphic to its dual, we conclude that the cohomology group in degree $k$ has the same dimension as the compactly supported cohomology group in degree $m-k$. For closed manifolds, this reads

$$\dim H^k(M,\mathbb{R})=\dim H^{m-k}(M,\mathbb{R}).$$

The isomorphism (23.2) is known as **Poincaré duality**. Its proof below uses the Poincaré Lemma, the Thom isomorphism, and an inductive argument based on the Mayer–Vietoris sequence and some homological algebra.

**Proof of Theorem 23.8.** For $M = \mathbb{R}^m$, the statement follows directly from the Poincaré Lemma and the Thom isomorphism.

Next, suppose $M = U_1 \cup U_2$, and assume that the statement of Theorem 23.8 holds for $U_1$, $U_2$, and their intersection $U_{12} = U_1 \cap U_2$. We aim to show that it then holds for $M$ as well.

For each $k$, consider the following commutative diagram:

$$\begin{tikzcd}
H^{k-1}(U_1,\mathbb{R}) \oplus H^{k-1}(U_2,\mathbb{R}) \arrow[r] \arrow[d] & H^{k-1}(U_{12},\mathbb{R}) \arrow[r] \arrow[d] & H^k(M,\mathbb{R}) \arrow[r] \arrow[d] & H^k(U_1,\mathbb{R}) \oplus H^k(U_2,\mathbb{R}) \arrow[r] \arrow[d] & H^k(U_{12},\mathbb{R}) \arrow[d] \\
H_c^{m-k+1}(U_1,\mathbb{R})^* \oplus H_c^{m-k+1}(U_2,\mathbb{R})^* \arrow[r] & H_c^{m-k+1}(U_{12},\mathbb{R})^* \arrow[r] & H_c^{m-k}(M,\mathbb{R})^* \arrow[r] & H_c^{m-k}(U_1,\mathbb{R})^* \oplus H_c^{m-k}(U_2,\mathbb{R})^* \arrow[r] & H_c^{m-k}(U_{12},\mathbb{R})^*
\end{tikzcd}$$

Here, the top row is the Mayer–Vietoris sequence for the standard de Rham cohomology, and the bottom row is the dual of the Mayer–Vietoris sequence for compactly supported de Rham cohomology. The vertical maps are the Poincaré duality isomorphisms from (23.2) for the respective open sets.

By assumption, the outer four vertical maps are isomorphisms. By the **Five Lemma** (cf. [**Wei94**, Lemma 1.3.4]), stated below, the middle vertical map is also an isomorphism.

**Lemma 23.9.** *Let*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f_1"] & B \arrow[r, "b"] \arrow[d, "f_2"] & C \arrow[r, "c"] \arrow[d, "f_3"] & D \arrow[r, "d"] \arrow[d, "f_4"] & E \arrow[d, "f_5"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & D' \arrow[r, "d'"] & E'
\end{tikzcd}$$

*be a commutative diagram in an abelian category (e.g., abelian groups or vector spaces over a field), with both rows exact. If $f_1$ is an epimorphism, $f_2$ and $f_4$ are isomorphisms, and $f_5$ is a monomorphism, then $f_3$ is also an isomorphism.*

Now suppose $M$ is a smooth, oriented manifold of finite type. Then there exists an open cover

$$M = \bigcup_{\alpha=1}^{N} U_\alpha$$

such that every nonempty finite intersection $U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}$ is diffeomorphic to $\mathbb{R}^m$. We prove Theorem 23.8 by induction on $N$.

The base case $N = 1$ is covered by the case $M = \mathbb{R}^m$. For the inductive step, let

$$W_1 = U_1, \qquad W_2 = \bigcup_{\alpha=2}^{N} U_\alpha, \qquad W_1 \cap W_2 = \bigcup_{\alpha=2}^{N} (U_\alpha \cap U_1).$$

By the induction hypothesis, Theorem 23.8 holds for $W_1$, $W_2$, and $W_1 \cap W_2$. Applying the Mayer–Vietoris argument above, it follows that the statement holds for $M = W_1 \cup W_2$.

$\square$

## Solutions to exercises

**Exercise 23.3.** By Theorem 23.1, we have $H^m(M') \cong \mathbb{R}$. By Stokes' Theorem, both $\int_M f^*\omega$ and $\int_{M'} \omega$ depend only on the cohomology class of $\omega$. Moreover, replacing $\omega$ with a constant multiple does not change the ratio that defines $\deg(f)$. Therefore, the formula for $\deg(f)$ is independent of the choice of $\omega$.

Suppose $q \in M'$ is a regular value of $f$ (which exists by Sard's Theorem). Since $\dim M = \dim M'$, for any $p \in f^{-1}(q)$, the differential $d_p f\colon T_pM \to T_qM'$ is an isomorphism. Therefore, on a sufficiently small neighborhood $U$ of $p$, the map $f$ is a diffeomorphism onto its image. In particular, the preimages of $q$ are isolated points. Since $M$ is compact, $f^{-1}(q)$ is a finite set:

$$f^{-1}(q) = \{p_1, \ldots, p_k\} \subset M.$$

Furthermore, we can choose a sufficiently small neighborhood $U'$ of $q$ such that

$$f^{-1}(U') = U_1 \cup \cdots \cup U_k$$

is a disjoint union of neighborhoods $U_i$ of $p_i$, each of which maps diffeomorphically onto $U'$ under $f$. By Theorem 23.1, there exists an $m$-form $\omega$ supported in $U'$ whose cohomology class generates $H^m(M') \cong \mathbb{R}$. For each $p_i$, let $\varepsilon_i = \pm 1$ depending on whether $d_{p_i} f\colon T_{p_i}M \to T_qM'$ is orientation-preserving or not.

We have

$$\int_M f^*\omega = \sum_{i=1}^k \int_{U_i} f^*\omega = \sum_{i=1}^k \varepsilon_i \int_{U'} \omega = \left(\sum_{i=1}^k \varepsilon_i\right) \int_{M'} \omega.$$

We conclude that

$$\deg(f) = \sum_{i=1}^k \varepsilon_i \in \mathbb{Z}. \tag{23.3}$$

In other words, $\deg(f)$ is the signed count of preimage points of a regular value $q$, with signs determined by comparing orientations. □

**Exercise 23.4.** In the context of holomorphic maps between closed holomorphic manifolds of the same dimension, the conclusion of the previous exercise takes a simpler form. Since holomorphic diffeomorphisms are always orientation-preserving (see the solution to Exercise 15.13), the degree of a holomorphic map is simply the actual number of preimage points of a regular value $q$.

We now show that every degree-$d$ polynomial

$$p(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_0$$

defines a holomorphic map $p\colon \mathbb{CP}^1 \to \mathbb{CP}^1$. It is then easy to see that its degree is $d$, because for generic $q \in \mathbb{C}$, the equation $p(z) = q$ has exactly $d$ solutions. As in Exercise 3.9, to extend the function $p\colon \mathbb{C} \to \mathbb{C}$ to $\mathbb{CP}^1$, we must specify the image of the added point $\infty = [0 : 1]$ and show that $p$ is holomorphic near that point. First, we simply define

$$p(\infty) = \infty.$$

Let $w$ denote the local coordinate near $\infty$, related to the coordinate $z$ on $\mathbb{C}$ by $w = 1/z$; see the solution to Exercise 3.9. With respect to the local coordinate $w$ on both the domain and the target, the function $p$ takes the new form

$$w \mapsto \frac{1}{p(1/w)} = \frac{w^d}{1 + a_{d-1}w + \ldots + a_0 w^d}.$$

For $w$ sufficiently close to 0, the denominator is nonzero, and the expression defines a well-defined holomorphic function. We conclude that $p\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ is holomorphic. $\square$

**Exercise 23.5.** Switching $C_1$ and $C_2$ corresponds to composing $f$ with the antipodal map $q \mapsto -q$ of $S^2$. The latter is an orientation preserving diffeomorphism. Therefore, it does not change the degree.

For the second part, it is useful to extend the definition to links which are disjoint union of embedded circles. Therefore, let each of $M_1$ and $M_2$ be a finite disjoint union of $S^1$, and let

$$\gamma_1\colon M_1 \longrightarrow \mathbb{R}^3 \quad \text{and} \quad \gamma_2\colon M_2 \longrightarrow \mathbb{R}^3$$

be embeddings with disjoint images $L_1$ and $L_2$. As before, define

$$f\colon M_1 \times M_2 \to S^2, \qquad (p_1, p_2) \mapsto \frac{\gamma_1(p_1) - \gamma_2(p_2)}{|\gamma_1(p_1) - \gamma_2(p_2)|}.$$

and let the **linking number** $\ell(L_1, L_2)$ of $L_1$ and $L_2$ to be the degree of the map $f$. It is clear that $\ell$ is additive in components of $L_1$ and $L_2$.

**Lemma 23.10.** *If $C_1$ bounds an oriented surface $\Sigma$ (such that the orientation on $C_1$ is the boundary orientation), then $\ell(L_1, L_2) = 0$.*

**Proof.** The map $f$ is the restriction to boundary of the well-defined smooth map

$$F\colon \Sigma \times M_2 \longrightarrow S^2, \quad (p, q) \mapsto \frac{\gamma_1(p_1) - \gamma_2(p_2)}{|\gamma_1(p_1) - \gamma_2(p_2)|}$$

If $\omega$ is a volume form of $S^2$, then $d\omega = 0$ and by Stokes' Theorem

$$0 = \int_{\Sigma \times M_2} F^* d\omega = \int_{\Sigma \times M_2} d(F^*\omega) = \int_{M_1 \times M_2} f^*\omega = \ell(L_1, L_2).$$

□

Moving to the general case, suppose $L_1$ bounds an embedded surface transverse to $L_2$. Let $\{p_1, \ldots, p_k\}$ denote the points of intersections of $\Sigma$ and $L_2$. Removing sufficiently small balls $B_i$ on $\Sigma$ centered at $p_i$ produces a new surface $\Sigma'$ with additional boundary components $\gamma_1, \ldots, \gamma_k$. Let $L_1' = L_1 \cup \gamma_1 \cup \cdots \cup \gamma_k$ where $\gamma_i$ are given boundary orientation of $\Sigma'$. By the Lemma above,

$$\ell(L_1', L_2) = 0.$$

Therefore,

$$\ell(L_1, L_2) = -\sum_i \ell(\gamma_i, L_2).$$

Therefore, it is enough to prove that $\ell(\gamma_i, L_2) = \pm 1$ depending on the sign of intersection of $\Sigma$ and $L_2$ and $p_i$. Note that the boundary orientation on $\gamma_i$ coming from $\gamma_i = \partial B_i$ is the opposite of the boundary orientation coming from $\gamma_i = \partial \Sigma'$. Therefore, the desired result follows from the following lemma.

**Lemma 23.11.** *Suppose $C_2 \subset \mathbb{R}^3$ is an oriented embedded circle, $p \in C_2$, and $D$ is a sufficiently small oriented disk intersecting $C_2$ transversely at $p$, with boundary $\gamma = \partial D$. Then*

$$\ell(\gamma, C_2) = \pm 1,$$

*depending on whether the direct sum orientation on $T_pD \oplus T_pC_2$ agrees with the standard orientation on $\mathbb{R}^3$ or is opposite to it.*

**Proof.** Without loss of generality, and after applying a linear transformation, we may assume that $p = 0 \in \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$, that

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq \varepsilon^2\} \times \{0\}$$

for some sufficiently small $\varepsilon > 0$, and that $C_2$ is transverse to the plane $z = 0$ with $T_0C_2 = \partial_z$. Choose a direction $v$ in

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \times \{0\}$$

such that the line $\mathbb{R} \cdot v$ intersects $C_2$ only at $0 \in \mathbb{R}^3$. Then, viewing $v$ as a point in $S^2$, we find that $f^{-1}(v)$ consists of a single point $p = (-\varepsilon v, 0) \in \gamma \times C_2$. It is easy to verify (using the transversality of $C_2$ to the plane $z = 0$) that $v$ is a regular value. It follows from (23.3) that $\deg(f) = \pm 1$, depending on whether $d_pf$ is orientation-preserving or reversing.

Now observe that the vector fields $\partial_\theta$ on $\gamma$ and $\partial_z$ on $C_2$ determine an oriented frame for $T_p(\gamma \times C_2)$. Under $d_pf$, this frame is mapped to $(\partial_\theta, \partial_z)$ at $v$, which is an oriented basis for $T_vS^2$. Therefore, $d_pf$ is orientation-preserving, and hence $\deg(f) = 1$. $\square$

$\square$

Chapter 24

# Flow of vector fields and Lie derivative

Ordinary differential equations (ODEs) arise naturally in the study of motion and change, with their origins rooted in classical mechanics. Newton's laws, for instance, describe how the position and velocity of a particle evolve over time, leading directly to second-order differential equations. More broadly, an ODE expresses how a quantity changes infinitesimally in response to another – often time. From a geometric point of view, ODEs form a bridge between vector fields and diffeomorphisms: a vector field gives a direction of motion at each point, while the solutions to the associated ODE – called integral curves – trace out the actual paths followed. In this sense, diffeomorphisms can be thought of as the "integrated" versions of vector fields, capturing how points move under their flow. We will use these diffeomorphisms to understand how tensors change along a vector field. This leads us to the notion of the Lie derivative.

**Definition 24.1.** Suppose $M$ is a smooth manifold and $X$ is a smooth vector field on $M$. The ordinary differential equation (ODE) associated to $X$ with initial value $p_0 \in M$ is given by

$$\dot{x}(t) = X(x(t)), \qquad x(0) = p_0,$$

where $\dot{x}(t)$ denotes the derivative of $x(t)$ with respect to the "time" variable $t$. A solution to this equation is a smooth curve $x\colon I \to M$, for some time interval $I \ni 0$, such that $x(0) = p_0$ and the tangent vector to the curve at $x(t)$ is equal to the vector $X(x(t))$. In other words, the curve follows the direction of the vector field $X$ at every point.

**Remark 24.2.** If the manifold has a boundary, the domain $I$ of a solution curve may be a closed or half-open interval. For manifolds without boundary, the maximal interval of existence is always an open subinterval of $\mathbb{R}$. To keep the discussion focused, we will sometimes consider only the boundaryless case below.

In local coordinates $x = (x_1, \ldots, x_m)$ on a neighborhood of $p_0$, the vector field $X$ takes the form

$$X(x) = \sum_{i=1}^{m} a_i(x)\, \partial_{x_i},$$

and the solution curve $x^{p_0}(t)$ starting at $p_0$ corresponds to a collection $(x_1(t), \ldots, x_m(t))$ of $m$ smooth functions satisfying the system of equations

$$\dot{x}_i = a_i(x(t)) \qquad \text{for all } i = 1, \ldots, m.$$

Thus, calculations can be carried out locally in $\mathbb{R}^m$, and as we move along the integral curve, we can transition from one coordinate chart to another to cover its entire domain. As a result, many results about ODEs on $\mathbb{R}^m$ naturally extend to smooth manifolds with little extra work. However, the nontrivial topology of a manifold can lead to interesting long-term behaviors that make the theory richer and more subtle than in Euclidean space.

The following is the fundamental existence and uniqueness theorem in the theory of ODEs; c.f. [**Lee13**, Theorems 9.11 and 9.15].

**Theorem 24.3.** (1) *Suppose $M$ is a smooth manifold without boundary. Then for every point $p_0 \in M$, there exists a maximal open interval $I = I_{p_0} = (-a, b)$, with $a, b \in \mathbb{R}_+ \cup \{\infty\}$, on which the solution $x^{p_0} \colon (-a, b) \to M$ to the ODE*

$$\dot{x}(t) = X(x(t)), \qquad x(0) = p_0$$

*is defined and unique. As $t$ approaches $-a$ or $b$, the solution "escapes to infinity" in the sense that it eventually leaves every compact subset of $M$. If $M$ has a boundary, the solution may hit the boundary in finite time, in which case the maximal interval of existence may be half-open or closed.*

(2) *The endpoints $(a, b) = (a(p_0), b(p_0))$ and the solution curve vary smoothly with the initial point $p_0$.*

Putting all initial points together (and assuming that $M$ has no boundary), the second statement implies that there exists an open subset $U \subset \mathbb{R} \times M$ on which the *flow* function

$$\Phi \colon U \to M, \qquad (t, p) \mapsto x^p(t)$$

is defined and smooth. In the special case where $M$ is closed (i.e., compact with no boundary), and in many other examples, we have $U = \mathbb{R} \times M$. This is the setting that will arise in most of the exercises below.

Originally, we fixed the initial point and let $t$ vary to obtain a curve. Now, we fix $t$ and allow the initial point $p$ to vary. This gives a map

$$\Phi_t \colon U_t := U \cap (\{t\} \times M) \to M,$$

where $U_t \subset M$ is the set of points for which the flow exists at time $t$. It follows from the uniqueness of solutions to ODEs that

$$\Phi_t \circ \Phi_s = \Phi_{t+s}$$

on the domain where both sides are defined.

**Remark 24.4.** If $U_t = M$, then $\Phi_t$ is a diffeomorphism of $M$, with inverse $\Phi_{-t}$. More generally, as $t \to 0$, the sets $U_t$ expand and exhaust $M$; that is,

$$M = \bigcup_{t>0} U_t.$$

Therefore, in the limiting definitions that follow, we may treat $\Phi_t$ as though it is defined on all of $M$.

**Exercise 24.5.** Find the solutions to the ODE

$$\dot{x} = Ax$$

on $\mathbb{R}^m$, where $A$ is a constant $m \times m$ matrix. Also, find the solutions to the ODE

$$\dot{x} = x^2$$

on $\mathbb{R}$. For which (if either) of these two equations are the solutions defined for all time?

**Exercise 24.6.** Compute the flow of the vector field

$$X = \sum_{i=1}^{n} x_i \partial y_i$$

on $\mathbb{R}^{2n}$ with coordinates $(\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n))$.

**Exercise 24.7.** Suppose $X$ is a smooth vector field on a smooth manifold $M$ and there exists a smooth function $f \colon M \longrightarrow \mathbb{R}$ such that $X \cdot f = df(X) > 0$. Show that the flow of $X$ has no periodic orbit (i.e., an integral curve $\gamma \colon \mathbb{R} \to M$ such that $\gamma(t+T) = \gamma(t)$ for some fixed $T > 0$ and all $t \in \mathbb{R}$). Show that the flow of any gradient vector field $\nabla f$ has no non-constant periodic orbit.

**Exercise 24.8.** Let $X$ be a smooth vector field on a manifold $M$ and $f \colon M \longrightarrow \mathbb{R}_{\pm}$ be a smooth function. Show every integral curve of $fX$ is a reparametrization of an integral curve of $X$. Prove that there is $f \colon M \longrightarrow \mathbb{R}_{+}$ such that every integral curve of $fX$ is defined over the entire $\mathbb{R}$.

We can use the diffeomorphisms $\Phi_t$ to push forward vector fields (via $d\Phi_t$) or pull back differential forms (by composing with $d\Phi_t$), and study how these objects change as $t$ varies. This leads to a notion of differentiation along the flow of $X$ for vector fields and differential forms, known as the *Lie derivative*. For vector fields, we show that this derivative coincides with the Lie bracket defined in Corollary 7.12. From this perspective, the Lie bracket describes the extent to which the flows of two vector fields fail to commute. We also prove an explicit expression for the Lie derivative of differential forms, known as *Cartan's formula*.

**Definition 24.9.** Suppose $\Phi_t$ is the flow of a vector field $X$ on a manifold $M$. For another vector field $Y$ on $M$, the **Lie derivative** of $Y$ along $X$ is the vector field

$$\underset{\rightarrow}{L}_X Y = \lim_{t\to 0} \frac{(\Phi_t)_* Y - Y}{t},$$

which is defined at every point $x \in M$ by Remark 24.4. Similarly, for any differential form $\eta$ on $M$, the **Lie derivative** of $\eta$ along $X$ is the differential form

$$L_X \eta = \lim_{t\to 0} \frac{(\Phi_t)^* \eta - \eta}{t},$$

which has the same degree as $\eta$. In particular, for a differential 0-form, that is, a smooth function $f\colon M \to \mathbb{R}$, we have

$$\begin{aligned}(L_X f)(p) &= \lim_{t\to 0} \frac{(\Phi_t)^* f - f}{t}\Big|_p = \lim_{t\to 0} \frac{f\circ\Phi_t(p) - f(p)}{t} \\ &= \lim_{t\to 0} \frac{f(x^p(t)) - f(x^p(0))}{t} = df(\dot{x}^p(0)) \\ &= d_p f(X(p)) = X\cdot f\big|_p;\end{aligned} \tag{24.1}$$

i.e., the Lie derivative of a function is simply the derivative of the function in the direction of the vector field.

**Remark 24.10.** In the definition of the Lie derivative of a vector field, the arrow placed above $L$ is nonstandard notation. We have included it to remind the reader that push-forward with respect to $\Phi_t$ is being used. In fact, as we will explain in the next section, to maintain compatibility with the Lie derivative of differential forms, it is desirable to pull back vector fields as well. Since $\Phi_{-t}$ is the inverse of $\Phi_t$, pulling back via $\Phi_t$ corresponds to pushing forward via $\Phi_{-t}$. Therefore, we may define

$$\underset{\leftarrow}{L}_X Y = \lim_{t\to 0} \frac{(\Phi_{-t})_* Y - Y}{t}.$$

This is indeed the definition found in many sources. Since $\Phi_{-t}$ is the flow of $-X$, it follows that

$$\underset{\leftarrow}{L}_X Y = -\underset{\rightarrow}{L}_X Y.$$

**Theorem 24.11.** *Suppose $\Phi_t$ is the flow of a vector field $X$ on a manifold $M$. For any other vector field $Y$ on $M$, we have*

$$\underset{\rightarrow}{L}_X Y = [Y, X] \quad \text{(or equivalently } \underset{\leftarrow}{L}_X Y = [X, Y]\text{)}$$

*Also, for any differential form $\eta$ on $M$ we have*

$$L_X \eta = d\big(\iota_X \eta\big) + \iota_X\big(d\eta\big).$$

**Proof.** Since $[Y, X]$ is defined as the commutator of the derivations corresponding to $X$ and $Y$, to prove the first relation, it suffices to show that both sides act identically on any smooth function $f\colon M \to \mathbb{R}$. We compute:

$$\begin{aligned} ((\underset{\rightarrow}{L}_X Y) \cdot f)\,|_p &= \left( \left( \lim_{t \to 0} \frac{(\Phi_t)_* Y - Y}{t} \right) \cdot f \right) \Bigg|_p \\ &= \lim_{t \to 0} \frac{(Y \cdot (f \circ \Phi_t))(\Phi_{-t}(p)) - (Y \cdot f)(p)}{t}. \end{aligned} \tag{24.2}$$

By (7.1), there exists a function $g(t, p)$ such that

$$f \circ \Phi_t(p) = f(p) + t g(t, p)$$

with $g(0, p) = \frac{\partial}{\partial t}(f \circ \Phi_t)\big|_{t=0} = (X \cdot f)(p)$. Substituting this expansion into (24.2), we get:

(24.3)

$$\begin{aligned} ((\underset{\rightarrow}{L}_X Y) \cdot f)\,|_p &= \lim_{t \to 0} \frac{(Y \cdot f)(\Phi_{-t}(p)) - (Y \cdot f)(p)}{t} + \lim_{t \to 0} (Y \cdot g(t, p))(\Phi_{-t}(p)) \\ &= (-X \cdot Y \cdot f + Y \cdot X \cdot f)(p) = ([Y, X] \cdot f)(p). \end{aligned}$$

This proves the first statement.

To prove Cartan's formula, we compare the Lie derivative operator $L_X$ with the operator $d \circ \iota_X + \iota_X \circ d$. We show that these two operators agree on all differential forms by verifying the following:

(1) Both operators satisfy the Leibniz rule:

$$L_X(\eta_1 \wedge \eta_2) = L_X(\eta_1) \wedge \eta_2 + \eta_1 \wedge L_X(\eta_2),$$

and similarly for $d \circ \iota_X + \iota_X \circ d$.

(2) They agree on functions, i.e., differential 0-forms:

$$L_X f = \iota_X(df) = X \cdot f.$$

(3) They agree on 1-forms $\eta$:

$$L_X \eta = d(\iota_X \eta) + \iota_X(d\eta).$$

Since smooth functions and 1-forms generate the algebra of all differential forms under wedge product, and both operators satisfy the Leibniz rule, it follows that

$$L_X = d \circ \iota_X + \iota_X \circ d$$

on all differential forms.

Item (2) was already established in (24.1). To prove that both operators satisfy the Leibniz rule, we compute:

(24.4)

$$\begin{aligned} L_X(\eta_1 \wedge \eta_2) &= \lim_{t\to 0} \frac{(\Phi_t)^*(\eta_1 \wedge \eta_2) - \eta_1 \wedge \eta_2}{t} = \lim_{t\to 0} \frac{(\Phi_t)^*\eta_1 \wedge (\Phi_t)^*\eta_2 - \eta_1 \wedge \eta_2}{t} \\ &= \lim_{t\to 0} \left( \frac{(\Phi_t)^*\eta_1 - \eta_1}{t} \wedge \eta_2 + \eta_1 \wedge \frac{(\Phi_t)^*\eta_2 - \eta_2}{t} \right) \\ &= L_X\eta_1 \wedge \eta_2 + \eta_1 \wedge L_X\eta_2. \end{aligned}$$

So $L_X$ satisfies the Leibniz rule.

We now verify that $d \circ \iota_X + \iota_X \circ d$ satisfies the same rule. First, recall:

$$\iota_X(\eta_1 \wedge \eta_2) = \iota_X\eta_1 \wedge \eta_2 + (-1)^{\deg \eta_1} \eta_1 \wedge \iota_X\eta_2.$$

Applying $d$ gives:

$$\begin{aligned} d \circ \iota_X(\eta_1 \wedge \eta_2) &= d(\iota_X\eta_1 \wedge \eta_2 + (-1)^{\deg \eta_1} \eta_1 \wedge \iota_X\eta_2) \\ &= d(\iota_X\eta_1) \wedge \eta_2 + (-1)^{\deg \iota_X\eta_1} \iota_X\eta_1 \wedge d\eta_2 \\ &\quad + (-1)^{\deg \eta_1} \left[ d\eta_1 \wedge \iota_X\eta_2 + (-1)^{\deg \eta_1} \eta_1 \wedge d(\iota_X\eta_2) \right]. \end{aligned}$$

Now compute $\iota_X \circ d(\eta_1 \wedge \eta_2)$. Since

$$d(\eta_1 \wedge \eta_2) = d\eta_1 \wedge \eta_2 + (-1)^{\deg \eta_1} \eta_1 \wedge d\eta_2,$$

we get

$$\begin{aligned} \iota_X \circ d(\eta_1 \wedge \eta_2) &= \iota_X(d\eta_1 \wedge \eta_2 + (-1)^{\deg \eta_1} \eta_1 \wedge d\eta_2) \\ &= \iota_X(d\eta_1) \wedge \eta_2 + (-1)^{\deg d\eta_1} d\eta_1 \wedge \iota_X\eta_2 \\ &\quad + (-1)^{\deg \eta_1} \left[ \iota_X\eta_1 \wedge d\eta_2 + (-1)^{\deg \eta_1} \eta_1 \wedge \iota_X(d\eta_2) \right]. \end{aligned}$$

Adding the two expressions gives:

$$(d \circ \iota_X + \iota_X \circ d)(\eta_1 \wedge \eta_2) = (d \circ \iota_X + \iota_X \circ d)(\eta_1) \wedge \eta_2 + \eta_1 \wedge (d \circ \iota_X + \iota_X \circ d)(\eta_2),$$

so this operator also satisfies the Leibniz rule.

Finally, to prove item (3), we verify directly in local coordinates that the two operators agree on 1-forms.

Choose local coordinates $(x_1, \ldots, x_m)$, and write

$$X = \sum_i a_i(x)\, \partial_{x_i}, \qquad \eta = \sum_j b_j(x)\, dx_j.$$

By linearity and the Leibniz rule, it suffices to check the case where $\eta = dx_j$.

On one hand,

$$(d \circ \iota_X + \iota_X \circ d)(dx_j) = d(\iota_X dx_j) = d(a_j).$$

On the other hand,

$$\begin{aligned} L_X(dx_j) &= \lim_{t\to 0} \frac{(\Phi_t)^* dx_j - dx_j}{t} = \lim_{t\to 0} \frac{d(x_j \circ \Phi_t) - dx_j}{t} \\ &= \lim_{t\to 0} d\left(\frac{x_j \circ \Phi_t - x_j}{t}\right) = d(\dot{x}_j) = d(a_j). \end{aligned}$$

Hence, both operators agree on 1-forms. $\square$

**Example 24.12.** If $M$ is an oriented $m$-manifold equipped with a volume form $\omega$, then

$$L_X\omega = d(\iota_X\omega) = \mathrm{Div}_\omega(X)\,\omega,$$

where $\mathrm{Div}_\omega(X)$ denotes the divergence of $X$ with respect to $\omega$, as defined earlier in Section 19. If $\mathrm{Div}_\omega(X) = 0$, this means that the flow $\Phi_t$ of $X$ preserves the volume of any $m$-dimensional region, although it may distort its shape.

**Exercise 24.13.** Let

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

For a volume form

$$\omega = f\,dx \wedge dy$$

on $\mathbb{R}^2$, show that $L_X\omega = 0$ iff $f$ is a function of distance from the origin.

**Exercise 24.14.** Suppose $\omega$ is a 2-form on a manifold $M$. We say that $\omega$ is *non-degenerate* if $\omega(u, \cdot) = 0$ implies $u = 0$ for all $u \in T_pM$. In other words, the map

$$T_pM \to T_p^*M, \quad u \mapsto u^\flat := \omega(u, \cdot)$$

is an isomorphism. (This is the skew-symmetric analogue of a Riemannian metric.)

Now suppose $\omega$ is closed and non-degenerate. Such a form is called a *symplectic form*. By the isomorphism above, for every smooth function $h\colon M \to \mathbb{R}$, there exists a unique vector field $X$ defined by

$$\iota_X\omega = -dh.$$

(1) Show that $L_X\omega = 0$.
(2) Show that $h$ is constant along integral curves of $X$. (The vector field $X$ is called the *Hamiltonian vector field* associated to the Hamiltonian function $h$, and the ODE defined by $X$ is called a *Hamiltonian system*.)

(3) Show that the action

$$S^1 \times \mathbb{C}^n \to \mathbb{C}^n, \quad (e^{it}, (z_1, \dots, z_n)) \mapsto (e^{it} z_1, \dots, e^{it} z_n)$$

is Hamiltonian with respect to the symplectic form

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n,$$

on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, where $x_i$ and $y_i$ are the real and imaginary parts of $z_i = x_i + \mathrm{i} y_i$. That is, the orbits of this action are integral curves of a Hamiltonian vector field.

**Remark 24.15.** Hamiltonian ODEs arise naturally in classical mechanics, where the function $h$ is typically the *energy* of the system (often the sum of kinetic and potential energy). The condition $\iota_X \omega = -dh$ encodes the equations of motion, and the fact that $L_X \omega = 0$ implies that the symplectic structure – and hence the phase space volume – is preserved under time evolution. The conservation of $h$ along integral curves of $X$ reflects the physical principle of energy conservation.

**Exercise 24.16.** Let $(x, y, z, w)$ be the standard coordinates of the Euclidean space $\mathbb{R}^4$. Let $X = \partial_z$ and $\omega = (xyz) dz \wedge dx$. Compute the Lie derivative $L_X \omega$.

**Exercise 24.17.** Let $m \geq 2$, and let $M$ be a smooth oriented $m$-manifold equipped with a volume form $\omega$. For any $(m-2)$-form $\eta \in \Omega^{m-2}(M)$, show that the equation

$$\iota_X \omega = d\eta$$

defines a unique vector field $X$ on $M$ associated to $\eta$. Show further that this vector field is *volume-preserving*, in the sense that the volume form $\omega$ is invariant under its flow.

What cohomological condition on $M$ is equivalent to the statement that *every* volume-preserving vector field arises from some $\eta$ in this way?

## Solutions to exercises

**Exercise 24.5.** If $m = 1$, then $A$ is a constant and the ODE

$$\dot{x} = Ax$$

can be rewritten as $\frac{d}{dt}\ln(x) = A$, which has the solution $x(t) = e^{tA}x(0)$ for any initial value $x(0)$. The same formula makes sense when $A$ is a matrix rather than just a number, where

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

The solutions are thus defined for all time $t$.

Similarly to the previous example, we can rewrite $\dot{x} = x^2$ as

$$\frac{d}{dt}(-x^{-1}) = 1,$$

which has the solution

$$-x^{-1}(t) = t - x^{-1}(0),$$

or

$$x(t) = \frac{x(0)}{1 - tx(0)}.$$

Clearly, as $t \to \frac{1}{x(0)}$, the solution approaches $\infty$. We conclude that for initial values $x(0) \neq 0$, the solutions are not defined for all time. For $x(0) = 0$, the solution $x(t)$ is constantly 0; i.e., $x = 0$ is a fixed point of the ODE flow $\Phi_t$. □

**Exercise 24.6.** The ODE equations of $X$ are

$$\dot{x}_i = 0 \quad \text{and} \quad \dot{y}_i = x_i \qquad \forall\ i = 1, \ldots, n.$$

Therefore, on any integral curve, the $x_i$-coordinates are fixed and the $y_i$-coordinates change linearly by $tx_i$; i.e.,

$$(\mathbf{x}(t), \mathbf{y}(t)) = (\mathbf{x}(0), \mathbf{y}(0) + t\mathbf{x}(0)).$$

□

**Exercise 24.7.** By the chain rule, the condition $X \cdot f = df(X) > 0$ means that $f$ is increasing along integral curves $\gamma$ of $X$. The only increasing functions on $S^1$ are constant functions. This can only occur if $\gamma$ is the constant map $\gamma(t) = p$, which implies $X(p) = 0$, and hence $p$ is a fixed point of the flow of $X$.

For any Riemannian metric $g$ on $M$, we have

$$df(\nabla f) = g(\nabla f, \nabla f) \geq 0,$$

with equality if and only if $\nabla f = 0$. The result now follows from the discussion above. $\square$

**Exercise 24.8.** Suppose
$$\gamma\colon I \to M$$
is an integral curve of $X$, i.e., $\dot{\gamma}(t) = X(\gamma(t))$ for all $t \in I$. Suppose
$$h\colon J \to I, \quad s \mapsto t = h(s),$$
is a reparametrization of $I$ by a smooth map, and define $\widetilde{\gamma}(s) = \gamma(h(s))$. By the chain rule,
$$\frac{d}{ds}\widetilde{\gamma}(s) = \left(\frac{d}{dt}\gamma\right)(h(s)) \cdot \frac{dh}{ds} = X(\widetilde{\gamma}(s)) \cdot \frac{dh}{ds}.$$
If
$$\frac{dh}{ds} = f(\widetilde{\gamma}(s)),$$
then $\widetilde{\gamma}$ is an integral curve of $fX$, showing that every integral curve of $fX$ is a reparametrization of an integral curve of $X$. To find the appropriate $h$, it is easier to determine its inverse, since the equation above implies
$$\frac{ds}{dt} = \frac{1}{\frac{dt}{ds}} = \frac{1}{f(\widetilde{\gamma}(s))}\bigg|_{s=h^{-1}(t)} = \frac{1}{f(\gamma(t))}.$$
Given the integral curve $\gamma(t)$, we find that
$$h^{-1}(t) = \int_0^t \frac{1}{f(\gamma(\tau))}\, d\tau$$
has the desired property.

For the second part, if $M$ is compact, then any smooth vector field on $M$ is complete, i.e., all its integral curves are defined for all time. In this case, taking $f = 1$ suffices. Thus, we may assume that $M$ is non-compact. We want to choose $f$ such that the range of $h^{-1}$ is all of $\mathbb{R}$ for each integral curve of $X$. In other words, as $p$ approaches the "infinity" of $M$, the function $f(p)$ must decay to zero sufficiently fast.

Let $\{K_n\}_{n\geq 0}$ be an exhaustion of $M$ by compact subsets such that
$$K_n \subset K_{n+1}^\circ \qquad \text{and} \qquad \bigcup_{n\geq 0} K_n = M.$$
For each $n$, choose a smooth function $\chi_n\colon M \to [0,1]$ such that $\chi_n \equiv 1$ on $K_n \setminus K_{n-1}^\circ$ and $\operatorname{supp}(\chi_n) \subset K_{n+1}^\circ \setminus K_{n-2}$. (Here we define $K_n = \emptyset$ for

$n < 0$.) We will use this to construct a Riemannian metric $g$ on $M$ with the following property: for any $n_0$,

$$\lim_{n\to\infty} \operatorname{dist}_g(K_{n_0}, M \setminus K_n^\circ) = \infty.$$

In other words, as $n \to \infty$, the length of any curve that starts in $K_{n_0}$ and exits $K_n$ must diverge.

Let $g_0$ be an arbitrary Riemannian metric on $M$. Set $c_0 = 1$. For each $n > 0$, choose $c_n > 0$ such that

$$\operatorname{dist}_{c_n g_0}(K_{n-1}, M \setminus K_n^\circ) = 1.$$

Then define

$$g = \left( \sum_{n=0}^{\infty} c_n \chi_n \right) g_0.$$

This sum converges because every point $p \in M$ lies in the support of at most three of the $\chi_n$. Moreover, $g$ satisfies the desired property because

$$\begin{aligned}\operatorname{dist}_g(K_{n_0}, M \setminus K_n^\circ) &\geq \sum_{\ell=n_0+1}^{n} \operatorname{dist}_g(K_{\ell-1}, M \setminus K_\ell^\circ) \\ &\geq \sum_{\ell=n_0+1}^{n} \operatorname{dist}_{c_\ell g_0}(K_{\ell-1}, M \setminus K_\ell^\circ) = n - n_0.\end{aligned}$$

With respect to the Riemannian metric $g$ defined above, let

$$g(p) = \sqrt{g(X(p), X(p))} \qquad \forall\, p \in M.$$

Let $f\colon M \to \mathbb{R}_+$ be any smooth function such that

$$f(p) \leq \frac{1}{g(p)},$$

or equivalently,

$$\frac{1}{f(p)} \geq g(p)$$

for all $p \in M$. Then the vector field $Y = fX$ satisfies $|Y(p)| \leq 1$ for all $p \in M$. Therefore, any integral curve of $Y$ defined over a finite interval $[a, b)$ or $(b, a]$ remains in some $K_n$. By Theorem 24.3, any maximal integral curve of $Y$ must be defined on all of $\mathbb{R}$. □

**Exercise 24.13.** We have

$$L_X\omega = d\iota_X\omega = -d\big(f(x,y)(ydy + xdx)\big) = \frac{-1}{2} d\big(f(r,\theta) r dr\big) = \frac{r}{2}\frac{\partial f}{\partial \theta} dr \wedge d\theta.$$

Therefore, $L_X\omega = 0$ iff $f$ is a function $r$. $\square$

**Exercise 24.14.** Since $\omega$ is closed, we have

$$L_X\omega = d\iota_X\omega = -d(dh) = 0.$$

We have

$$X \cdot h = dh(X) = -\omega(X, X) = 0.$$

Therefore, $h$ is constant along integral curves of $X$.

Differentiating the action with respect to $t$ we get

$$\frac{d}{dt}(e^{it}z_1, \ldots, e^{it}z_n) = \mathfrak{i}(e^{it}z_1, \ldots, e^{it}z_n).$$

We conclude that each $(e^{it}z_1, \ldots, e^{it}z_n)$ is an integral curve of the ODE

$$\dot{z} = X(z) = \mathfrak{i}\ z.$$

In real coordinates,

$$X(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} -y_i\partial_{x_i} + x_i\partial_{y_i}.$$

Therefore,

$$-\iota_X\omega = d\ \frac{1}{2}\sum_{i=1}^{n} x_i^2 + y_i^2.$$

We conclude that the action in the question is the hamiltonian ODE flow of the hamiltonian $h = \frac{1}{2}\sum_{i=1}^{n} x_i^2 + y_i^2$. $\square$

**Exercise 24.16.** We have

$$\begin{aligned} L_X\omega =& d\iota_{\partial_z}\omega + \iota_{\partial_z}d\omega = d\big(xyz\,dx\big) + \iota_{\partial_z}\big(xz\,dy \wedge dz \wedge dx\big) \\ & xy\,dz \wedge dx + xz\,dy \wedge dx - xz\,dy \wedge dx = xy\ dz \wedge dx. \end{aligned}$$

$\square$

**Exercise 24.17.** The map

$$\Gamma(M, TM) \longrightarrow \Omega^{m-1}(M), \qquad X \longmapsto \iota_X\omega$$

defines an isomorphism between the space of vector fields on $M$ and the space of differential $(m-1)$-forms (see the solution to Exercise 17.1). Therefore, for any $(m-2)$-form $\eta \in \Omega^{m-2}(M)$, there exists a unique vector field $X$ satisfying

$$\iota_X\omega = d\eta.$$

The Lie derivative of $\omega$ with respect to $X$ is

$$L_X\omega = d\iota_X\omega = d(d\eta) = 0.$$

Therefore, the volume form $\omega$ is invariant under the flow of $X$.

Conversely, suppose

$$L_X\omega = d\iota_X\omega = 0;$$

i.e., $\iota_X\omega$ is a closed form. If $H^{m-1}(M, \mathbb{R}) = 0$, then every closed $(m-1)$-form is exact, and there exists $\eta \in \Omega^{m-2}(M)$ such that $\iota_X\omega = d\eta$. This condition is both necessary and sufficient, since any closed $(m-1)$-form can be written as $\iota_X\omega$ for a unique vector field $X$.

Chapter 25

# Tensor fields and Lie derivative

In previous sections, we have studied various examples of tensors, such as vector fields, differential forms, and metrics. In general, given a finite-dimensional real vector space $V$, a **tensor of type** $(p, q)$ is an element

$$T \in \underbrace{V \times \cdots \times V}_{p \text{ copies}} \times \underbrace{V^* \times \cdots \times V^*}_{q \text{ copies}}.$$

Equivalently, a tensor is often defined as a multilinear map

$$T\colon \underbrace{V^* \times \cdots \times V^*}_{p \text{ copies}} \times \underbrace{V \times \cdots \times V}_{q \text{ copies}} \longrightarrow \mathbb{R}.$$

Examples of tensors on a vector space $V$:

- Type $(0, 0)$: Scalars (real numbers).
- Type $(1, 0)$: Vectors (elements of $V$).
- Type $(0, 1)$: Covectors or linear forms (elements of $V^*$).
- Type $(1, 1)$: Linear transformations (elements of $V \otimes V^* \cong \mathrm{Hom}(V, V)$).

We can then require additional properties. For instance, symmetric positive-definite $(0, 2)$-tensors correspond to inner products on $V$.

The above definition extends pointwise to any vector bundle $E \to M$, and in particular to the tangent bundle $TM$ of a smooth manifold.

**Definition 25.1.** A **tensor field** of type $(p, q)$ on $M$ is a smooth section of the bundle

$$TM^{\otimes p} \otimes T^*M^{\otimes q}.$$

Common examples of tensor fields on a manifold $M$:

- Type $(0,0)$: Smooth functions on $M$.
- Type $(1,0)$: Vector fields.
- Type $(0,k)$ and skew-symmetric: Differential $k$-forms.
- Type $(0,2)$, symmetric and positive-definite: Riemannian metrics.

In the previous section, we studied the Lie derivative of vector fields and differential forms and obtained explicit formulas in each case. A natural question is whether these two notions of Lie derivative are related.

In fact, the Lie derivative can be defined for arbitrary tensor fields, and it satisfies a product rule in the following sense: if $\tau_1$ is a tensor field of type $(p_1, q_1)$ and $\tau_2$ is of type $(p_2, q_2)$, then $\tau_1 \otimes \tau_2$ is a tensor of type $(p_1 + p_2, q_1 + q_2)$, and

$$L_X(\tau_1 \otimes \tau_2) = (L_X\tau_1) \otimes \tau_2 + \tau_1 \otimes (L_X\tau_2). \tag{25.1}$$

To define the Lie derivative for tensors of mixed type, one must apply the same operation – either pullback or push-forward – to both the $TM$ and $T^*M$ components. In order to preserve Cartan's formula, it is standard practice to use pullbacks: that is, to define the Lie derivative of a $(p,q)$-tensor, we apply $\Phi_t^* := (\Phi_{-t})_*$ to the $p$ vector field components and $\Phi_t^*$ to the $q$ covector components. In conclusion, the operator

$$L_X\tau = \lim_{t\to 0} \frac{\Phi_t^*\tau - \tau}{t}$$

is well-defined on all tensor fields. It recovers the identity $L_X\eta = (d \circ \iota_X + \iota_X \circ d)\eta$ for differential forms and $\underleftarrow{L}_X Y = [X, Y]$ for vector fields.

For the rest of this section, we adopt the pullback convention and simply write $L_X Y$ when referring to the Lie derivative of a vector field.

**Lemma 25.2.** *Lie derivative of tensors satisfies the Leibniz formula* (25.1).

**Proof.** As in (24.4), since pullback and tensor product commute, we have

$$\begin{aligned}
L_X(\tau_1 \otimes \tau_2) &= \lim_{t\to 0} \frac{\Phi_t^*(\tau_1 \otimes \tau_2) - (\tau_1 \otimes \tau_2)}{t} \\
&= \lim_{t\to 0} \frac{\Phi_t^*(\tau_1) \otimes \Phi_t^*(\tau_2) - \Phi_t^*(\tau_1) \otimes \tau_2 + \Phi_t^*(\tau_1) \otimes \tau_2 - \tau_1 \otimes \tau_2}{t} \\
&= \lim_{t\to 0} \frac{\Phi_t^*(\tau_1) \otimes (\Phi_t^*(\tau_2) - \tau_2) + (\Phi_t^*(\tau_1) - \tau_1) \otimes \tau_2}{t} \\
&= L_X(\tau_1) \otimes \tau_2 + \tau_1 \otimes L_X(\tau_2).
\end{aligned}$$

□

**Remark 25.3.** The general product formula above, and the scalar version we use below, are related by "taking the trace."

Given a $(p, 0)$-tensor $\tau_1$ and a $(0, p)$-tensor $\tau_2$, we can form their tensor product $\tau = \tau_1 \otimes \tau_2$, which is a $(p, p)$-tensor. The Lie derivative of $\tau$ satisfies the Leibniz rule (25.1), producing an equality of $(p, p)$-tensors.

On the other hand, contracting $\tau_1$ and $\tau_2$ yields a smooth function $f = \tau_2(\tau_1)$, whose Lie derivative also satisfies a product rule:

$$L_X f = (L_X \tau_2)(\tau_1) + \tau_2(L_X \tau_1).$$

These two formulas are related via the fact that contraction is compatible with the Lie derivative: taking the Lie derivative of a contraction is the same as contracting the Lie derivative.

In this sense, the scalar product rule arises from the tensor-level formula by "taking the trace." Our proof of this compatibility below is given in the specific context of vector fields contracting differential forms.

As a starting point, suppose $\eta$ is a differential 1-form, and $X$, $Y$ are vector fields. Then the function $f = \eta(X)$ is smooth, and we will prove that

$$df(Y) = Y \cdot f = L_Y f = (L_Y \eta)(X) + \eta(L_Y X).$$

Using the formulas from the previous section, we compute:

$$\begin{aligned} Y \cdot (\eta(X)) &= (d\iota_Y \eta)(X) + (\iota_Y d\eta)(X) - \eta([X, Y]) \\ &= X \cdot (\eta(Y)) + (d\eta)(Y, X) - \eta([X, Y]). \end{aligned}$$

Rearranging terms gives the following formula for $d\eta$ in terms of how it acts on vector fields:

$$d\eta(X, Y) = X \cdot (\eta(Y)) - Y \cdot (\eta(X)) - \eta([X, Y]).$$

In general, we obtain the following:

**Theorem 25.4.** *If $\eta$ is a $k$-form, then $d\eta$ is the $(k+1)$-form whose action on $k+1$ vector fields $X_0, \ldots, X_k$ is given by*

(25.2)

$$\begin{aligned} (d\eta)(X_0, \ldots, X_k) = &\sum_{j=0}^{k} (-1)^j X_j \cdot \eta(X_0, \ldots, \widehat{X_j}, \ldots, X_k) \\ &+ \sum_{i<j} (-1)^{i+j} \eta\big([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k\big). \end{aligned}$$

Here, we provide a computational proof based on calculations in local charts. A coordinate-free proof is also possible by applying the product rule and expanding $\mathcal{L}_{X_0}(\eta(X_1, \ldots, X_k))$, as we did above.

**Proof.** It is easy to verify that the equation is local, linear in each input, and skew-symmetric on both sides. So we only need to verify it for a single local expression
$$\eta = f(x)\,dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$
and $X_a = g_a(x)\,\partial_{x_{j_a}}$ for $a = 0, \ldots, k$. To eliminate the coefficients $g_a$ and simplify the task, we first prove that if (25.2) holds for $(X_0, \ldots, X_k)$, then it also holds for $(g(x)X_0, X_1, \ldots, X_k)$. By symmetry, the same is true if we modify other inputs.

Using the identity
$$[gX_0, X_i] = g[X_0, X_i] - (X_i \cdot g)\,X_0,$$
we compute:
$$\begin{aligned}
&(d\eta)(gX_0, X_1, \ldots, X_k) - g\,X_0 \cdot \eta(X_1, \ldots, X_k) \\
&\quad - \sum_{a=1}^{k}(-1)^a X_a \cdot \eta(gX_0, \ldots, \widehat{X_a}, \ldots, X_k) \\
&\quad - \sum_{a=1}^{k}(-1)^a \eta([gX_0, X_a], X_1, \ldots, \widehat{X_a}, \ldots, X_k) \\
&\quad - \sum_{1 \le a < b \le k} (-1)^{a+b} \eta([X_a, X_b], gX_0, \ldots, \widehat{X_a}, \ldots, \widehat{X_b}, \ldots, X_k) \\
&= g\Big((d\eta)(X_0, \ldots, X_k) - \sum_{a=0}^{k}(-1)^a X_a \cdot \eta(X_0, \ldots, \widehat{X_a}, \ldots, X_k) \\
&\quad - \sum_{0 \le a < b \le k} (-1)^{a+b} \eta([X_a, X_b], X_0, \ldots, \widehat{X_a}, \ldots, \widehat{X_b}, \ldots, X_k)\Big) \\
&\quad - \sum_{a=1}^{k}(-1)^a (X_a \cdot g)\,\eta(X_0, \ldots, \widehat{X_a}, \ldots, X_k) \\
&\quad + \sum_{b=1}^{k}(-1)^b (X_b \cdot g)\,\eta(X_0, \ldots, \widehat{X_b}, \ldots, X_k).
\end{aligned}$$
The last two terms cancel out, and we obtain the desired result.

Having reduced the problem to the case $X_a = \partial_{x_{j_a}}$ for all $a = 0, \ldots, k$, we compute:

(25.3)
$$\begin{aligned}
(d\eta)(X_0, \ldots, X_k) &= \sum_{i_0} \frac{\partial f}{\partial x_{i_0}}\,(dx_{i_0} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k})\,(\partial_{x_{j_0}}, \ldots, \partial_{x_{j_k}}) \\
&= \sum_{i_0 \in \{j_0, \ldots, j_k\}} \frac{\partial f}{\partial x_{i_0}}\,(dx_{i_0} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k})\,(\partial_{x_{j_0}}, \ldots, \partial_{x_{j_k}}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{a=0}^{k}(-1)^a X_a \cdot \eta(X_0,\ldots,\widehat{X_a},\ldots,X_k) \\
&\qquad + \sum_{a<b}(-1)^{a+b}\eta([X_a,X_b],X_0,\ldots,\widehat{X_a},\ldots,\widehat{X_b},\ldots,X_k) \\
&= \sum_{a=0}^{k}(-1)^a \partial_{x_{j_a}}\Big(f(x)\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}(\partial_{x_{j_0}},\ldots,\widehat{\partial_{x_{j_a}}},\ldots,\partial_{x_{j_k}})\Big) \\
&= \sum_{a=0}^{k}(-1)^{j_a}\frac{\partial f}{\partial x_{j_a}}\,(dx_{j_a}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k})\,(\partial_{x_{j_a}},\partial_{x_{j_0}},\ldots,\widehat{\partial_{x_{j_a}}},\ldots,\partial_{x_{j_k}}).
\end{aligned}$$

Moving $\partial_{x_{j_a}}$ forward to its correct position eliminates $(-1)^{j_a}$ and we get the same expression as (25.3). □

Differential 2-forms and metrics are both tensors of type $(0,2)$ – the former being skew-symmetric, the latter symmetric. Therefore, given a vector field $X$ and a Riemannian metric $g$ on a manifold $M$, the Lie derivative of $g$ with respect to $X$ is the symmetric (but not necessarily positive-definite) $(0,2)$-tensor

$$L_X g = \lim_{t\to 0}\frac{\Phi_t^* g - g}{t} \in \Gamma(M, \mathrm{Sym}^2(T^*M)). \tag{25.4}$$

where $\Phi_t$ is the flow of $X$. We say that $X$ is a **Killing vector field** if $L_X g = 0$. In other words, the flow of $X$ consists of isometries. We state, without proof, the following result, which follows from applying the product rule and expanding $\mathcal{L}_X(g(Y,Z))$ (see [**Lee13**, Proposition 12.15]).

**Theorem 25.5.** *Let $g$ be a Riemannian metric on a manifold $M$, and let $X, Y, Z$ be vector fields on $M$. Then*

$$(L_X g)(Y,Z) = X\cdot g(Y,Z) - g([X,Y],Z) - g([X,Z],Y).$$

**Exercise 25.6.** Recall that a Riemannian metric $g$ identifies the space of vector fields and 1-forms by sending a vector field $Y$ to the 1-form $Y^\flat = g(Y,-)$. Show that if $X$ is a Killing vector field, then this identification commutes with the Lie derivative; that is,

$$(L_X Y)^\flat = L_X(Y^\flat).$$

**Exercise 25.7.** Let $g$ be the standard Euclidean metric on $\mathbb{R}^2$, and let $X = x\partial_x + y\partial_y$. Compute $L_X g$.

**Exercise 25.8.** An operator $D$ acting on sections of a vector bundle $E$ over a smooth manifold $M$ (such as tensor fields) is called **tensorial** if it is $C^\infty(M)$-linear; that is,

$$D(f\xi) = fD(\xi) \quad \text{for all } f \in C^\infty(M), \xi \in \Gamma(E).$$

By first showing that the operators

$$L_X \circ L_Y - L_Y \circ L_X - L_{[X,Y]} \quad \text{and} \quad \iota_{[X,Y]} - L_X \circ \iota_Y + \iota_Y \circ L_X$$

are tensorial, prove the identities

$$[L_X, L_Y] := L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]} \quad \text{and} \quad \iota_{[X,Y]} = [L_X, \iota_Y] := L_X \circ \iota_Y - \iota_Y \circ L_X.$$

## Solutions to exercises

**Exercise 25.6.** We show that $(L_X Y)^\flat$ and $L_X(Y^\flat)$ act identically on any vector field $Z$. By definition,

$$(L_X Y)^\flat(Z) = g(L_X Y, Z) = g([X, Y], Z).$$

On the other hand, using the product rule,

$$(L_X(Y^\flat))(Z) = L_X(Y^\flat(Z)) - Y^\flat(L_X Z) = L_X(g(Y, Z)) - g(Y, [X, Z]).$$

Therefore, the identity we seek to verify is

$$g([X, Y], Z) = L_X(g(Y, Z)) - g(Y, [X, Z]),$$

which is equivalent to

$$X \cdot g(Y, Z) = L_X(g(Y, Z)) = g([X, Y], Z) + g([X, Z], Y).$$

The latter equality follows from Theorem 25.5 and the assumption that $L_X g = 0$.

□

**Exercise 25.7.** Using Theorem 25.5, we find the symmetric $(0, 2)$-tensor $L_X g$ by computing its action on basis vector fields. There are three terms to calculate.

We have

$$(L_X g)(\partial_x, \partial_x) = X \cdot g(\partial_x, \partial_x) + 2g([\partial_x, X], \partial_x).$$

The first term on the right-hand side is zero, and

$$[\partial_x, X] = [\partial_x, x\partial_x + y\partial_y] = [\partial_x, x\partial_x] + [\partial_x, y\partial_y] = \partial_x.$$

Therefore,

$$(L_X g)(\partial_x, \partial_x) = 2.$$

By symmetry of the problem, we similarly have

$$(L_X g)(\partial_y, \partial_y) = 2.$$

Finally,

$$(L_X g)(\partial_x, \partial_y) = g([\partial_x, X], \partial_y) + g([\partial_y, X], \partial_x) = 0,$$

since $g(\partial_x, \partial_y) = 0$.

We conclude that

$$L_X g = 2g.$$

This can also be verified directly from the definition (25.4). The flow of the radial vector field $X$ is

$$\Phi_t(x, y) = e^t(x, y).$$

Therefore,

$$\Phi_t^* g = \Phi_t^*(dx \otimes dx + dy \otimes dy) = e^{2t} g.$$

We conclude that

$$L_X g = \frac{d}{dt}\big(e^{2t} g\big)\Big|_{t=0} = 2g.$$

$\square$

**Exercise 25.8** Suppose $T$ is a type $(r,s)$ tensor field and $f \in C^\infty(M)$. We will use the Leibniz rule for Lie derivatives,

$$L_X(fT) = (X \cdot f)T + f L_X T,$$

to compute

$$[L_X, L_Y](fT) = L_X(L_Y(fT)) - L_Y(L_X(fT)).$$

Using the Leibniz rule we obtain

$$\begin{aligned} L_X(L_Y(fT)) &= L_X((Y \cdot f)T + f L_Y T) \\ &= X \cdot (Y \cdot f)T + (Y \cdot f)L_X T + X \cdot f\, L_Y T + f L_X L_Y T. \end{aligned}$$

Similarly,

$$L_Y(L_X(fT)) = Y \cdot (X \cdot f)T + (X \cdot f)L_Y T + Y \cdot f\, L_X T + f L_Y L_X T.$$

Subtracting these, many terms cancel, and we find:

$$[L_X, L_Y](fT) = f[L_X, L_Y]T + (X \cdot (Y \cdot f) - Y \cdot (X \cdot f))T.$$

On the other hand,

$$L_{[X,Y]}(fT) = ([X,Y] \cdot f)T + f L_{[X,Y]} T.$$

So the difference

$$\big([L_X, L_Y] - L_{[X,Y]}\big)(fT) = f([L_X, L_Y] - L_{[X,Y]})(T)$$

is $C^\infty(M)$-linear, hence tensorial.

Similarly, for the second identity, note that the interior product $\iota_Y$ is tensorial:

$$\iota_Y(fT) = f \iota_Y T.$$

Then,

$$L_X(\iota_Y(fT)) = L_X(f \iota_Y T) = (X \cdot f)\iota_Y T + f L_X \iota_Y T,$$

$$\iota_Y(L_X(fT)) = \iota_Y((X \cdot f)T + f L_X T) = (X \cdot f)\iota_Y T + f \iota_Y L_X T.$$

Subtracting gives:

$$[L_X, \iota_Y](fT) = f[L_X, \iota_Y](T),$$

so $[L_X, \iota_Y]$ is tensorial. Since $\iota_{[X,Y]}$ is also tensorial, their difference is tensorial:

$$\big(\iota_{[X,Y]} - [L_X, \iota_Y]\big)(fT) = f\big(\iota_{[X,Y]} - [L_X, \iota_Y]\big)(T).$$

Since the differences $[L_X, L_Y] - L_{[X,Y]}$ and $\iota_{[X,Y]} - [L_X, \iota_Y]$ are tensorial, they are determined pointwise. It therefore suffices to verify that they vanish on constant tensors at a point.

To simplify things further, we show that both $[L_X, L_Y] - L_{[X,Y]}$ and $\iota_{[X,Y]} - [L_X, \iota_Y]$ satisfy the Leibniz rule.

Let $T_1, T_2$ be tensor fields. Using the product rule for the Lie derivative repeatedly, we get

$$\begin{aligned}
[L_X, L_Y](T_1 \otimes T_2) &= L_X(L_Y(T_1 \otimes T_2)) - L_Y(L_X(T_1 \otimes T_2)) \\
&= L_X((L_Y T_1) \otimes T_2 + T_1 \otimes (L_Y T_2)) \\
&\quad - L_Y((L_X T_1) \otimes T_2 + T_1 \otimes (L_X T_2)) \\
&= (L_X L_Y T_1) \otimes T_2 + (L_Y T_1) \otimes L_X T_2 \\
&\quad + L_X T_1 \otimes L_Y T_2 + T_1 \otimes L_X L_Y T_2 \\
&\quad - (L_Y L_X T_1) \otimes T_2 - (L_X T_1) \otimes L_Y T_2 \\
&\quad - L_Y T_1 \otimes L_X T_2 - T_1 \otimes L_Y L_X T_2 \\
&= ([L_X, L_Y] T_1) \otimes T_2 + T_1 \otimes ([L_X, L_Y] T_2).
\end{aligned}$$

On the other hand,

$$L_{[X,Y]}(T_1 \otimes T_2) = (L_{[X,Y]} T_1) \otimes T_2 + T_1 \otimes (L_{[X,Y]} T_2).$$

We conclude that $[L_X, L_Y] - L_{[X,Y]}$ satisfies the Leibniz rule.

The proof for $\iota_{[X,Y]} - [L_X, \iota_Y]$ is similar and left to the reader.

In conclusion, we have shown that

$$[L_X, L_Y] = L_{[X,Y]} \quad \text{and} \quad [L_X, \iota_Y] = \iota_{[X,Y]}$$

are tensorial and satisfy the Leibniz rule. Since the differences vanish on constant vector fields and differential 1-forms (check in your own), they vanish identically. □

Chapter 26

# Straightening Theorem

Every vector field $X$ is locally of the form

$$X = \sum_{i=1}^{m} a_i(x) \frac{\partial}{\partial x_i}$$

for some smooth coefficient functions $a_i$. A natural and useful question is whether there exist local coordinates in which all the coefficients $a_i$ are constant. This would imply that their partial derivatives vanish, greatly simplifying many calculations. If such coordinates exist (and $X$ is nontrivial), one can further perform a linear change of variables to obtain local coordinates in which

$$X = \frac{\partial}{\partial x_1}.$$

The question becomes more challenging and interesting when one considers not just one vector field, but several vector fields, and asks whether there exist local coordinates in which all of them take the form

$$X_i = \frac{\partial}{\partial x_i}$$

for $i = 1, \ldots, k$. Surprisingly, this problem has an elegant solution known as the *Straightening Theorem* (also called the Flow-box Theorem or the Frobenius Theorem in the integrable distribution case).

The proof of the Straightening Theorem relies on the relationship between the Lie bracket $[X, Y]$ of two vector fields and the commutativity of their flows $\Phi_t^X$ and $\Phi_t^Y$. More precisely, for each point $p \in M$ and for sufficiently small $t$, consider the path

$$\gamma(t) = \Phi_{-t}^Y \circ \Phi_{-t}^X \circ \Phi_t^Y \circ \Phi_t^X(p).$$

This path is constant (meaning $\gamma(t) \equiv p$) if and only if the flows $\Phi_t^X$ and $\Phi_t^Y$ commute. Moreover, as we will establish in exercises,

$$\dot{\gamma}(0) = 0 \quad \text{and} \quad \frac{1}{2}\,\frac{d^2}{dt^2}\gamma(t)\bigg|_{t=0} \tag{26.1}$$

is exactly the Lie bracket $[X, Y](p)$. Therefore, the vanishing of the Lie bracket $[X, Y] = 0$ characterizes the local commutativity of the flows, which is a key ingredient in straightening vector fields simultaneously.

**Theorem 26.1.** *The flows $\Phi_t^X$ and $\Phi_s^Y$ of $X$ and $Y$ commute, i.e.,*

$$\Phi_s^Y \circ \Phi_t^X \cong \Phi_t^X \circ \Phi_s^Y \quad \forall\ s, t$$

*if and only if $[X, Y] = 0$.*

**Proof.** It is a simple yet critical observation that if $\varphi\colon M \to M$ is a diffeomorphism, then the ODE flows $\Phi_t^X$ and $\Phi_t^{\varphi_* X}$ are related by

$$\Phi_t^{\varphi_* X} = \varphi \circ \Phi_t^X \circ \varphi^{-1}.$$

Therefore, since $\Phi_{-s}^Y = (\Phi_s^Y)^{-1}$, the composition

$$\Phi_s^Y \circ \Phi_t^X \circ \Phi_{-s}^Y$$

is the ODE flow of $(\Phi_s^Y)_* X$. Hence, the commutativity of $\Phi_t^X$ and $\Phi_s^Y$ is equivalent to

$$\Phi_t^X = \Phi_t^{(\Phi_s^Y)_* X} \quad \text{for all } s, t.$$

Since two flows are equal if and only if their generating vector fields are equal, this means

$$X = (\Phi_s^Y)_* X \quad \text{for all } s.$$

If this holds for sufficiently small $s$, then by Theorem 24.11,

$$[X, Y] = \lim_{s \to 0} \frac{(\Phi_s^Y)_* X - X}{s} = 0.$$

Conversely, suppose $[X, Y] = 0$. For every point $p \in M$, define a path $\gamma(s)$ in the vector space $T_pM$ by

$$\gamma(s) = (\Phi_s^Y)_*\big(X(\Phi_{-s}^Y(p))\big).$$

Thanks to the linear structure on $T_pM$, we have

$$\begin{aligned}
\dot{\gamma}(s) &= \lim_{h \to 0} \frac{\gamma(s+h) - \gamma(s)}{h} \\
&= \lim_{h \to 0} \frac{(\Phi_{s+h}^Y)_*\big(X(\Phi_{-s-h}^Y(p))\big) - (\Phi_s^Y)_*\big(X(\Phi_{-s}^Y(p))\big)}{h} \\
&= (\Phi_s^Y)_* \lim_{h \to 0} \frac{(\Phi_h^Y)_*\big(X(\Phi_{-h}^Y(q))\big) - X(q)}{h},
\end{aligned}$$

where $q = \Phi^Y_{-s}(p)$. But by the definition of the Lie bracket,

$$\lim_{h\to 0} \frac{(\Phi^Y_h)_*\big(X(\Phi^Y_{-h}(q))\big) - X(q)}{h} = [X,Y](q) = 0.$$

Therefore, $\dot\gamma(s) = 0$, so $\gamma(s)$ is constant and equal to $\gamma(0) = X(p)$. This implies

$$X = (\Phi^Y_s)_* X \quad \text{for all } s,$$

as desired. □

**Exercise 26.2.** For any $p \in M$, choose local coordinates around $p$ so that $p$ corresponds to $0 \in \mathbb{R}^m$, and write the vector fields $X, Y$ locally as

$$X(x) = \sum_{i=1}^m a_i(x)\partial_{x_i}, \quad Y(x) = \sum_{i=1}^m b_i(x)\partial_{x_i}.$$

Then the Taylor expansion of the flow of $X$ near 0 up to second order in $t$ is

$$\Phi^X_t(x) = x + tX(x) + \frac{t^2}{2}(DX)(x)\cdot X(x) + O(t^3),$$

where $(DX)(x)$ is the Jacobian matrix of the coefficient functions of $X$ at $x$, acting on vectors. The same holds for $Y$. Use this to prove that for

$$\gamma(t) := \Phi^Y_{-t}\circ\Phi^X_{-t}\circ\Phi^Y_t\circ\Phi^X_t(p),$$

we have

$$\gamma(t) = p + t^2[X,Y](p) + O(t^3).$$

**Theorem 26.3** (Straightening Theorem)**.** *Suppose $X_1,\dots,X_k$ are vector fields on a smooth manifold $M$, and let $p \in M$. Further, assume that*

*(1) The vector fields $X_1,\dots,X_k$ are linearly independent on an open neighborhood of $p$,*

*(2) The vector fields commute pairwise; that is, $[X_i,X_j] = 0$ on an open neighborhood of $p$ for all $1 \le i,j \le k$.*

*Then there exist local coordinates $(x_1,\dots,x_m)$ defined on a sufficiently small neighborhood of $p$ such that*

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1,\dots,k.$$

**Proof.** It is simple linear algebra that we can choose local coordinates $(y_1,\dots,y_m)$ near $p$ such that $p$ corresponds to $0 \in \mathbb{R}^m$ and $X_i(0) = \frac{\partial}{\partial y_i}$ for $i = 1,\dots,k$.

Define a map $\varphi$ on a sufficiently small neighborhood of $0 \in \mathbb{R}^m$ by

$$(y_1,\dots,y_m) = \varphi(x_1,\dots,x_m) := \Phi^{X_1}_{x_1}\circ\cdots\circ\Phi^{X_k}_{x_k}(0,\dots,0,x_{k+1},\dots,x_m).$$

It is straightforward to check that $d_0\varphi = \mathrm{id}_{\mathbb{R}^m}$; hence, $\varphi$ is a local diffeomorphism fixing the origin.

Moreover, for each $i = 1, \dots, k$, using the commutativity of the flows (which follows from $[X_i, X_j] = 0$), we compute:

$$\begin{aligned}
\varphi_*(\partial_{x_i})|_{y=\varphi(x)} &= d_x\varphi(\partial_{x_i}) \\
&= \frac{d}{dt}\bigg|_{t=0} \Phi^{X_1}_{x_1} \circ \cdots \circ \Phi^{X_i}_{x_i+t} \circ \cdots \circ \Phi^{X_k}_{x_k}(0, \dots, 0, x_{k+1}, \dots, x_m) \\
&= \frac{d}{dt}\bigg|_{t=0} \Phi^{X_i}_t \left(\Phi^{X_1}_{x_1} \circ \cdots \circ \Phi^{X_i}_{x_i} \circ \cdots \circ \Phi^{X_k}_{x_k}(0, \dots, 0, x_{k+1}, \dots, x_m)\right) \\
&= \frac{d}{dt}\bigg|_{t=0} \Phi^{X_i}_t(y) = X_i(y).
\end{aligned}$$

This completes the proof. $\square$

**Exercise 26.4.** Consider the vector fields

$$X = f(x)\,\partial_y \quad \text{and} \quad Y = g(y)\,\partial_x$$

on $\mathbb{R}^2$. Find necessary and sufficient conditions on the pair $(f, g)$ such that there is a diffeomorphism $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with

$$\varphi_*(\partial_x) = X \quad \text{and} \quad \varphi_*(\partial_y) = Y.$$

## Solutions to exercises

**Exercise 26.2.** Without loss of generality, take $p = 0 \in \mathbb{R}^m$.

Using the given expansions,

$$\Phi_t^X(0) = 0 + tX(0) + \frac{t^2}{2}(DX)(0) \cdot X(0) + O(t^3),$$
$$\Phi_t^Y(x) = x + tY(x) + \frac{t^2}{2}(DY)(x) \cdot Y(x) + O(t^3).$$

Step 1: Compute $\Phi_t^Y \circ \Phi_t^X(0)$:

$$\Phi_t^Y(\Phi_t^X(0)) = \Phi_t^X(0) + tY(\Phi_t^X(0)) + \frac{t^2}{2}(DY)(\Phi_t^X(0)) \cdot Y(\Phi_t^X(0)) + O(t^3).$$

Expand $Y(\Phi_t^X(0))$ near 0:

$$Y(\Phi_t^X(0)) = Y(0) + (DY)(0) \cdot \big(tX(0) + O(t^2)\big) + O(t^2) = Y(0) + t(DY)(0)X(0) + O(t^2).$$

Similarly,

$$(DY)(\Phi_t^X(0)) \cdot Y(\Phi_t^X(0)) = (DY)(0)Y(0) + O(t).$$

Thus,

$$\Phi_t^Y \circ \Phi_t^X(0) = t(X(0) + Y(0)) + \frac{t^2}{2}\big((DX)(0)X(0) + 2(DY)(0)X(0) + (DY)(0)Y(0)\big) + O(t^3).$$

Step 2: Apply $\Phi_{-t}^X$ to the above:

$$\Phi_{-t}^X(z) = z - tX(z) + \frac{t^2}{2}(DX)(z) \cdot X(z) + O(t^3).$$

Expand $X(z)$ around 0 with

$$z = t(X(0) + Y(0)) + O(t^2),$$

so

$$X(z) = X(0) + (DX)(0)z + O(t^2) = X(0) + t(DX)(0)(X(0) + Y(0)) + O(t^2).$$

Similarly,

$$(DX)(z) \cdot X(z) = (DX)(0)X(0) + O(t).$$

Hence,

$$\Phi_{-t}^X \circ \Phi_t^Y \circ \Phi_t^X(0) = tY(0) + \frac{t^2}{2}\big((DY)(0)Y(0) + 2(DY)(0)X(0) - (DX)(0)Y(0) - (DX)(0)X(0)\big) + O(t^3).$$

Step 3: Apply $\Phi_{-t}^Y$:

$$\Phi_{-t}^Y(w) = w - tY(w) + \frac{t^2}{2}(DY)(w) \cdot Y(w) + O(t^3).$$

Expand $Y(w)$ near 0:
$$Y(w) = Y(0) + (DY)(0)w + O(t^2),$$
with
$$w = tY(0) + O(t^2).$$
Then,
$$Y(w) = Y(0) + t(DY)(0)Y(0) + O(t^2).$$
So,
$$\Phi^Y_{-t} \circ \Phi^X_{-t} \circ \Phi^Y_t \circ \Phi^X_t(0) = 0 + t^2\big((DY)(0)X(0) - (DX)(0)Y(0)\big) + O(t^3).$$
It can be easily checked that
$$[X, Y](0) = (DY)(0)X(0) - (DX)(0)Y(0).$$
Therefore,
$$\gamma(t) = \Phi^Y_{-t} \circ \Phi^X_{-t} \circ \Phi^Y_t \circ \Phi^X_t(0) = 0 + t^2[X, Y](0) + O(t^3),$$
which completes the proof. □

**Exercise 26.4.** For such a map $\varphi$ to exist, it is necessary that $[X, Y] = 0$. We compute:
$$\begin{aligned}
[X, Y] &= [f(x)\,\partial_y,\, g(y)\,\partial_x] \\
&= f(x)[\partial_y, g(y)\,\partial_x] - g(y)\,\partial_x(f(x))\,\partial_y \\
&= f(x)\,\frac{dg(y)}{dy}\,\partial_x - g(y)\,\frac{df(x)}{dx}\,\partial_y.
\end{aligned}$$
Therefore, $[X, Y] = 0$ if and only if
$$f(x)\,\frac{dg(y)}{dy} = 0 \quad \text{and} \quad g(y)\,\frac{df(x)}{dx} = 0.$$
If $\frac{df(x)}{dx} \neq 0$ for some $x$, then we must have $g \equiv 0$, which solves the equation above but does not satisfy $\varphi_*(\partial_y) = Y$. Hence, $\frac{df(x)}{dx} \equiv 0$. Similarly, we must have $\frac{dg(y)}{dy} \equiv 0$.

We conclude that $X$ and $Y$ must be constant vector fields:
$$X = a\,\partial_y \quad \text{and} \quad Y = b\,\partial_x$$
for some constants $a, b \neq 0$. Then, the map $\varphi(x, y) = (by, ax)$ satisfies $\varphi_*(\partial_x) = X$ and $\varphi_*(\partial_y) = Y$, as desired.

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